Spin-2 KK Mode Scattering in the Truncated RS1

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February 6th, 2020

MICHIGAN STATE UNIVERSITY

UC San Diego

What am I presenting?

- **Research I performed with:** R. Sekhar Chivukula, Kirtimaan A. Mohan, Dipan Sengupta, and Elizabeth H. Simmons

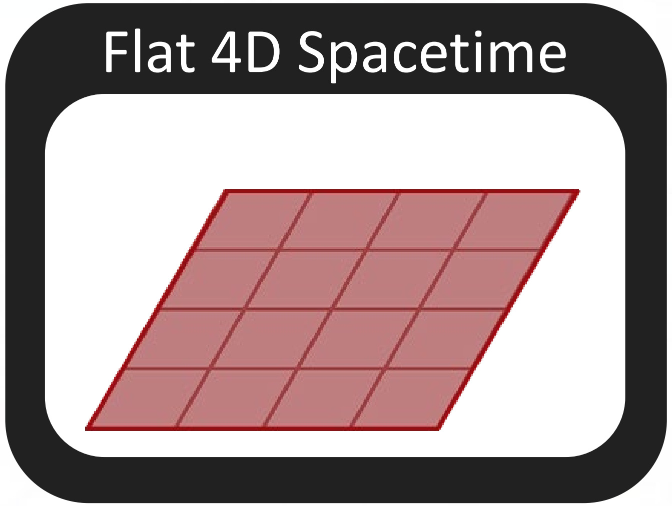
- **Our Relevant Papers:**
  - "Sum Rules for Massive Spin-2 Kaluza-Klein Elastic Scattering Amplitudes" [arXiv:1910.06159], Phys. Rev. D 100, 115033
  - "Scattering Amplitudes of Massive Spin-2 Kaluza-Klein States Grow Only as O(s)" [arXiv:1906.11098]

** COMING SOON! **

  - "Scattering Amplitudes and Strong Coupling Scales in Extra-Dimensional Theories" [arXiv: 2002.XXXXX]

The Overall Idea

4D Spacetime → 4D Graviton

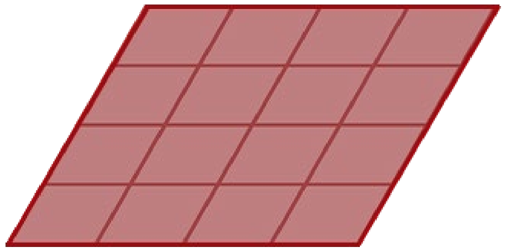


$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$ds^2 = dt^2 - d\vec{x}^{\,2} = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

4D Spacetime → 4D Graviton

Flat 4D Spacetime



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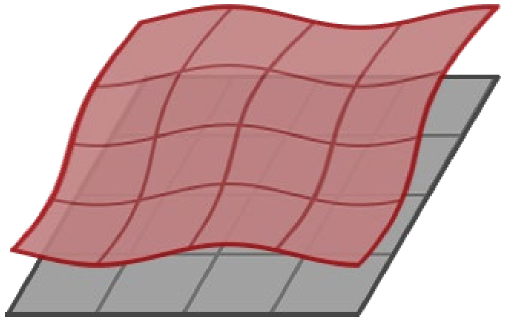
Nearly Flat Spacetime



$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\frac{2}{M_{\rm Pl}}\, \hat{h}^{(0)}_{\mu\nu}}_{\text{disturbance}}$$

M_{Pl} = the **reduced Planck mass**

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

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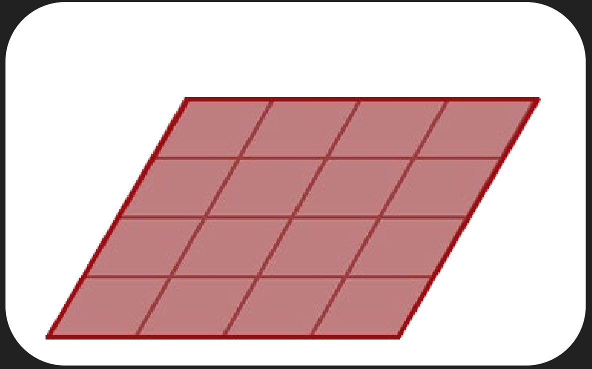

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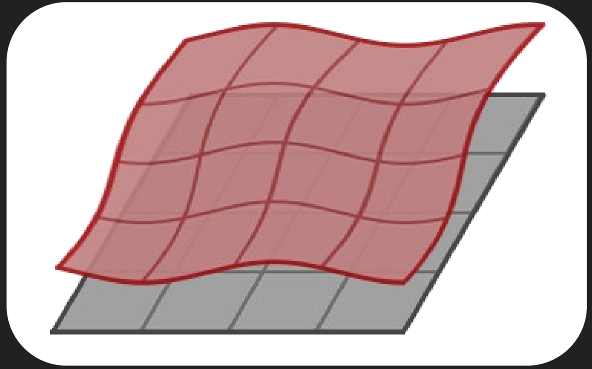

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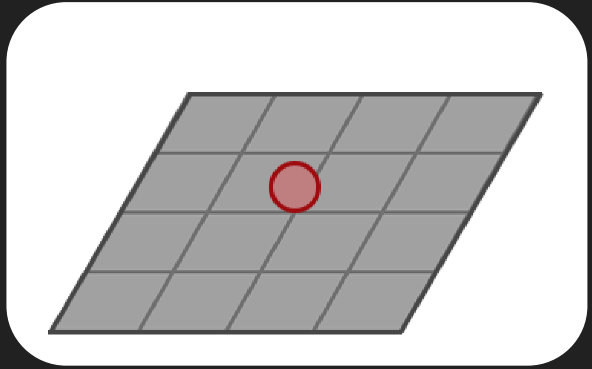
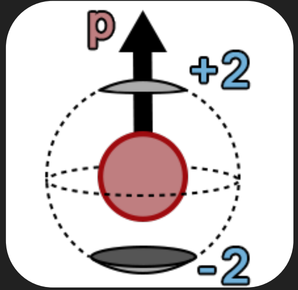
Graviton



$$\hat{h}^{(0)}_{\mu\nu} = \text{4D massless spin-2 field}$$
$$\equiv \text{the “4D graviton” field}$$

2 Possible Helicities:



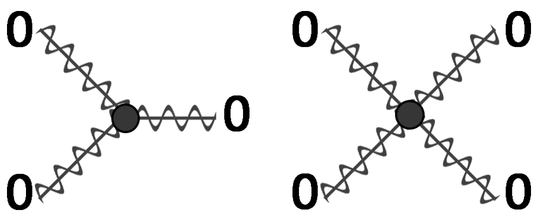
2-to-2 Scattering in 4DG

$$\mathcal{L}_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\sqrt{g}\,R(g)$$

Einstein-Hilbert Lagrangian

WFE

$$\boxed{g = \eta + \frac{2}{M_{\rm Pl}}\hat{h}^{(0)}}$$



Vertices

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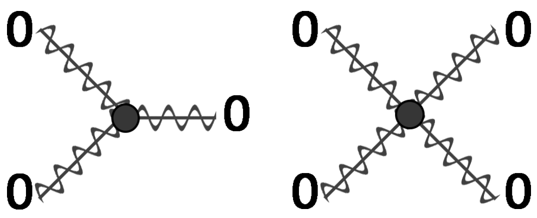
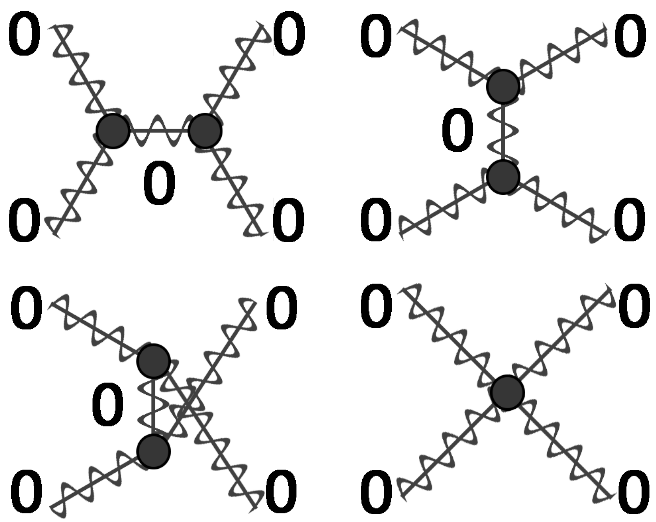


Vertices



Tree-Level 2-to-2 Diagrams

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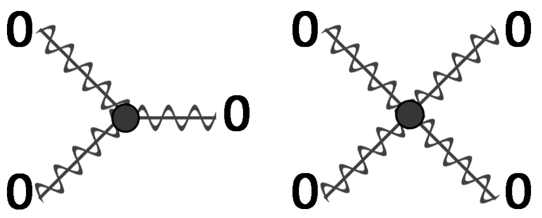
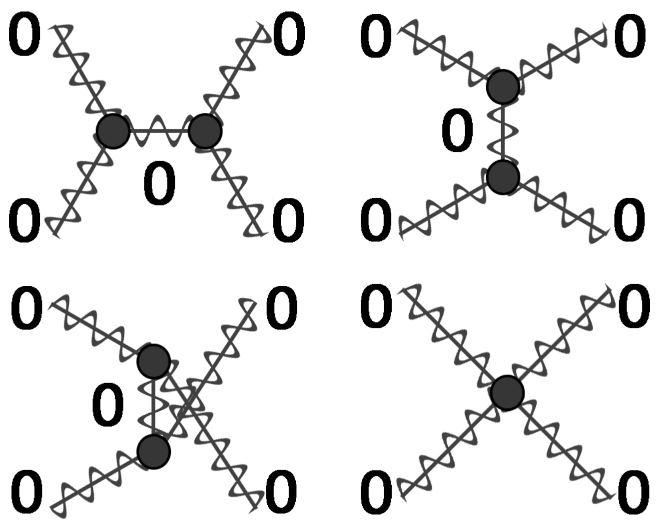
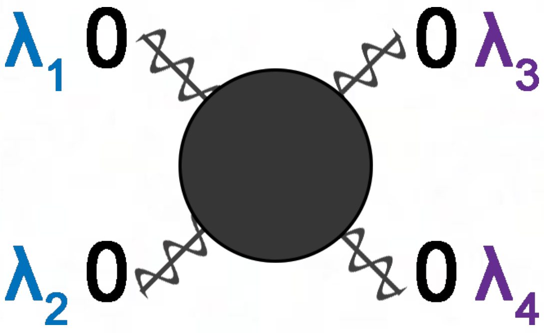


Vertices



Tree-Level 2-to-2 Diagrams

$\mathcal{M} =$



$$\propto \boxed{\left(\frac{E}{M_{\rm Pl}}\right)^2}$$

2^4 = 16 helicity combos

Because...

- **M_{Pl}** is the only dimensionful parameter.
- The 4D matrix element is dimensionless.

2-to-2 Scattering in FPG

$$\mathcal{L}_{\mathrm{FPG}} = \frac{2}{\tilde{\kappa}^2}\sqrt{g}\,R(g) + \boxed{\underbrace{\frac{m_n^2}{2}\left[(\hat{h}^{(n)}_{\mu\nu})^2 - \hat{h}^{(n)\,2}\right]}_{\text{Fierz-Pauli Mass Term}}}$$

$$g = \eta + \underline{\tilde{\kappa}}\,\hat{h}^{(n)}$$

Fierz-Pauli Massive Gravity

5 Possible Helicities:

Fierz Pauli massive gravity (FPG) is constructed by adding a Fierz-Pauli mass term to 4D gravity, which explicitly breaks 4D diffeomorphism invariance.

The field $h^{(n)}$ describes a **massive** spin-2 particle.

2-to-2 Scattering in FPG

$$\mathcal{L}_{\text{FPG}} = \frac{2}{\tilde{\kappa}^2}\sqrt{g}\,R(g) + \underbrace{\frac{m_n^2}{2}\left[(\hat{h}^{(n)}_{\mu\nu})^2 - \hat{h}^{(n)\,2}\right]}_{\text{Fierz-Pauli Mass Term}}$$

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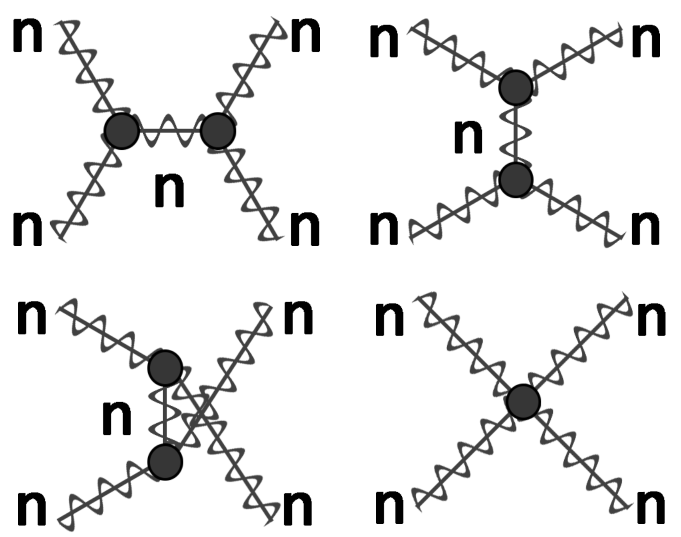
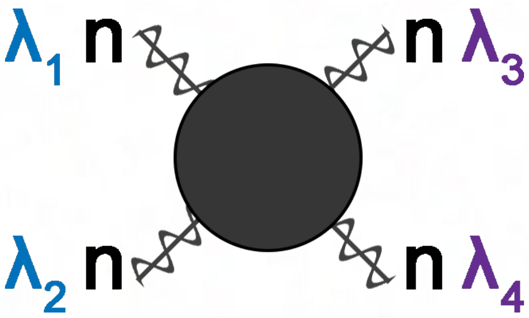


Tree-Level 2-to-2 Diagrams

5 Possible Helicities

$\mathcal{M} =$



5^4 = 625 helicity combos

2-to-2 Scattering in FPG

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Fierz-Pauli Massive Gravity

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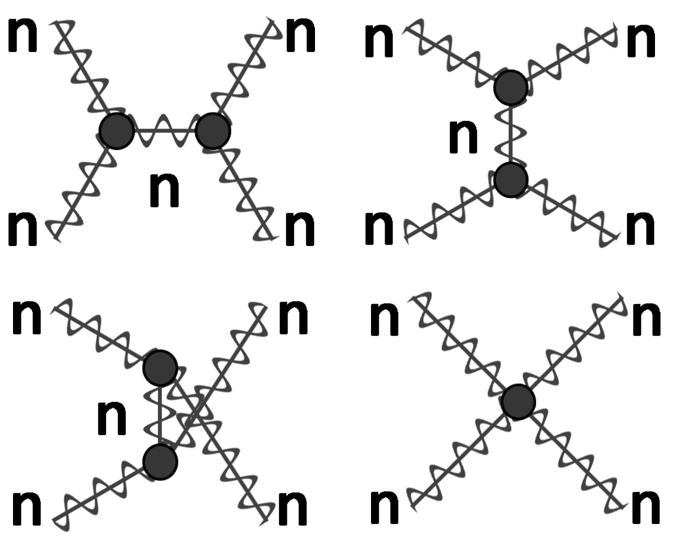
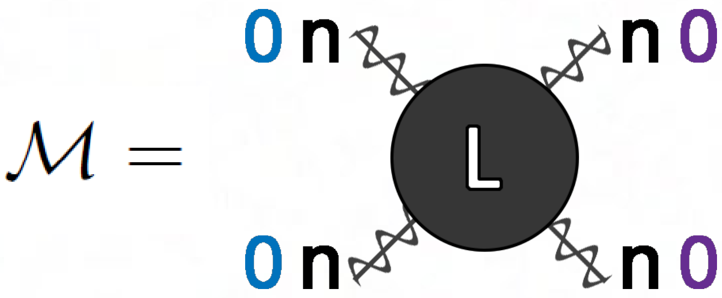


Tree-Level 2-to-2 Diagrams

5 Possible Helicities

<u>**The "Longitudinal" Process**</u>



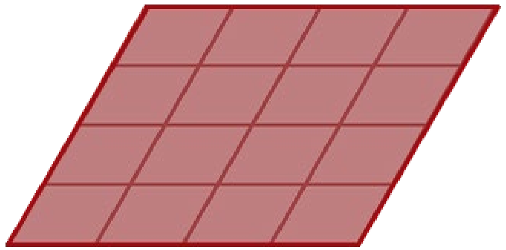
$$\boxed{\sim \frac{\tilde{\kappa}^2 E^{10}}{m_n^8}}$$

The helicity (0,0)-to-(0,0) process experiences **the fastest energy growth.**

Without 4D diffeomorphism protecting it, the matrix element grows rapidly.

Example: 5D Spacetime → 5D Graviton

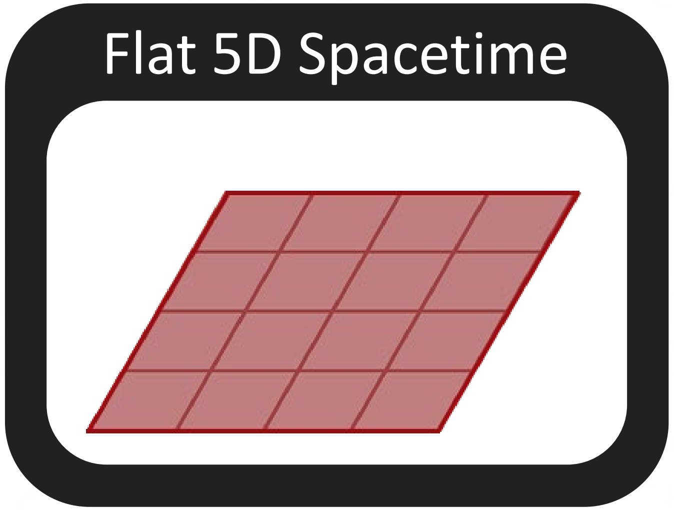
Flat 5D Spacetime

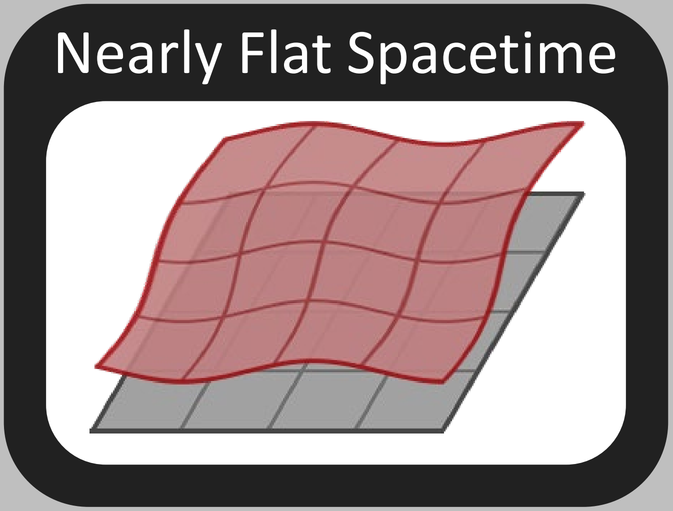


$$\eta_{MN} = \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}$$

Suppose new extra dimension has finite length $\boldsymbol{\pi r_c}$ where $\mathbf{r_c}$ = the **compactification radius**

Example: 5D Spacetime → 5D Graviton





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Suppose new extra dimension has finite length πr_c where r_c = the **compactification radius**

$$G_{MN} = \eta_{MN} + \underbrace{\kappa \hat{h}_{MN}}_{\text{disturbance}}$$

κ = the **5D coupling strength** and has units $[E]^{-3/2}$

$$ds^2 = G_{MN}\, dx^M\, dx^N$$

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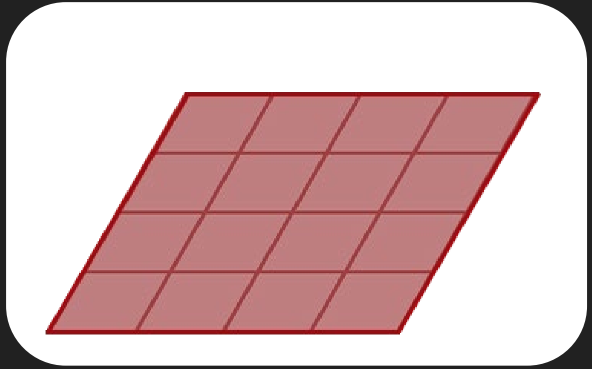

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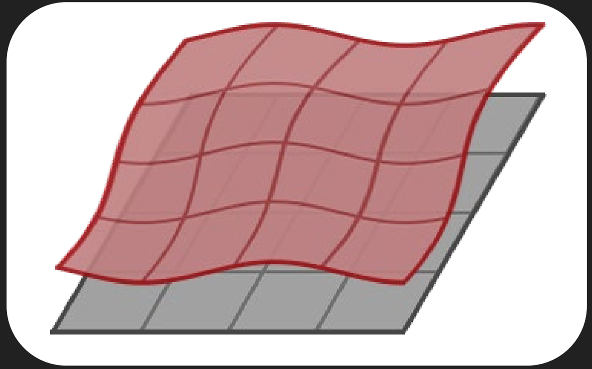
Nearly Flat Spacetime



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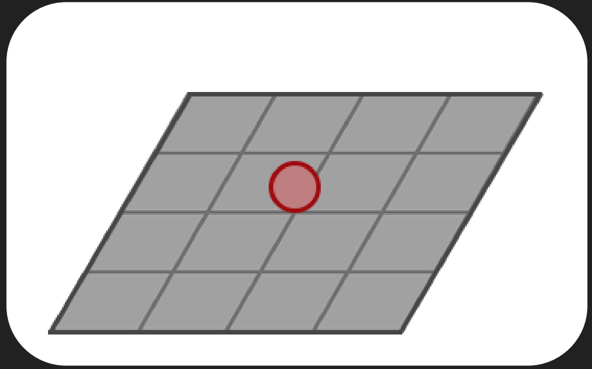
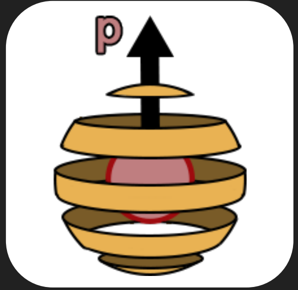
Graviton

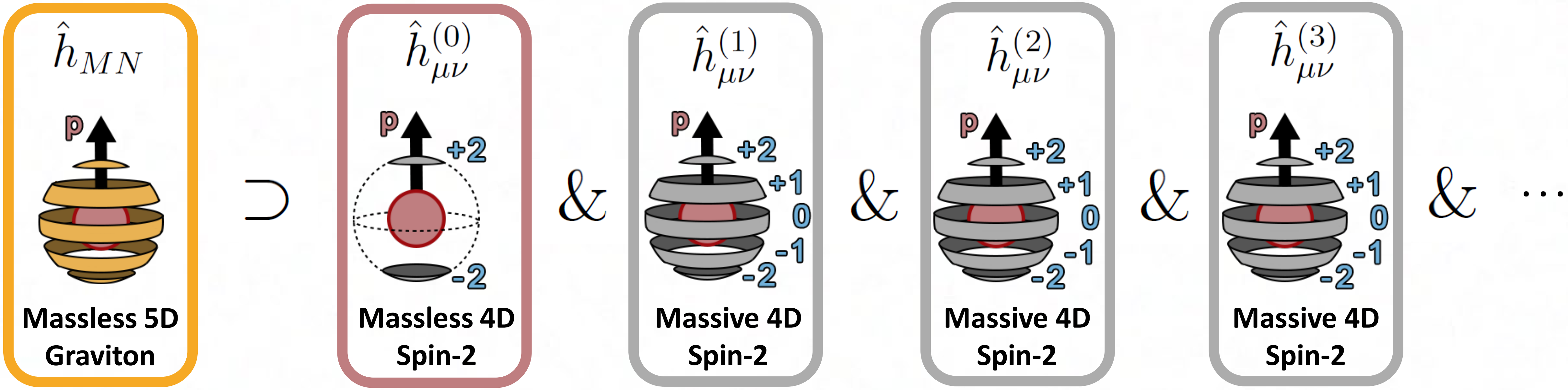


$\hat{h}_{MN} \equiv$ the "5D graviton" field

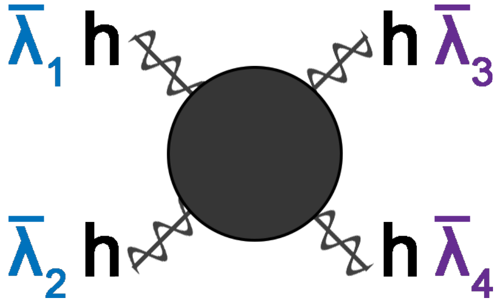
5 Possible Helicities:



Kaluza-Klein Decomposition: Hope for Massive Spin-2?

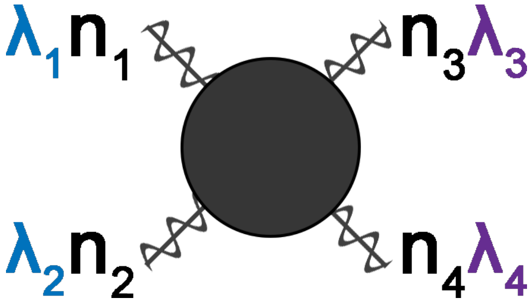


High-Energy 5D Behavior



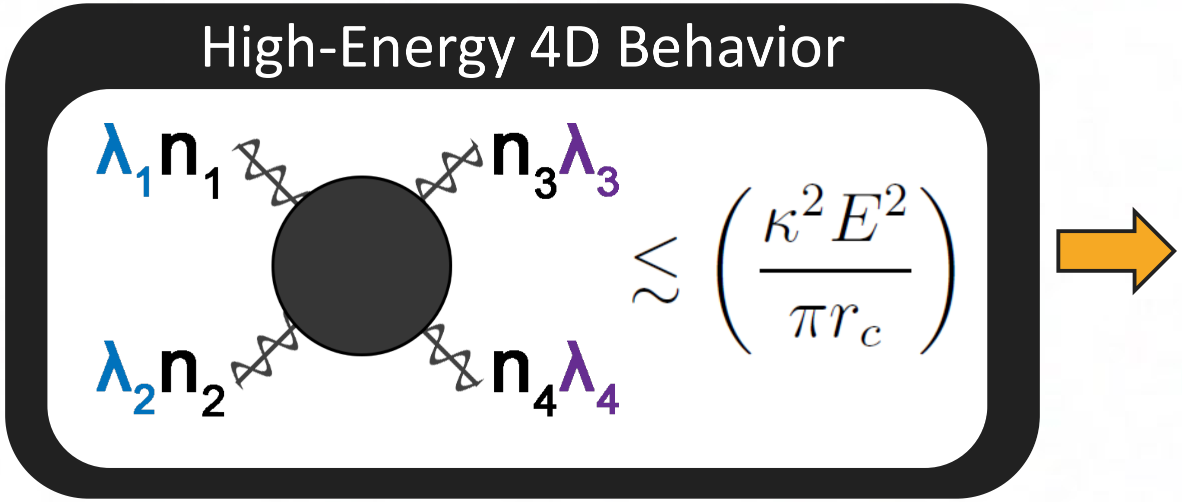
$$\sim \left(\kappa^2 E^3\right)$$

High-Energy 4D Behavior



$$\lesssim \left(\frac{\kappa^2 E^2}{\pi r_c}\right)$$

The "Punchlines" Slide



- We examine how this constraint impacts the **5D Randall-Sundrum (RS1) model**, which possesses warping in the extra dimension.
- We perform the full calculation, i.e. we don't rely on the "large warping" limit.
- We derive **sum rules** relating couplings & masses between KK modes.
- Cancellations of E^{10} behavior require infinitely many modes. **When truncating the KK sum, many states may be needed to sufficiently approximate the full matrix element.**

The Randall-Sundrum “RS1” Model

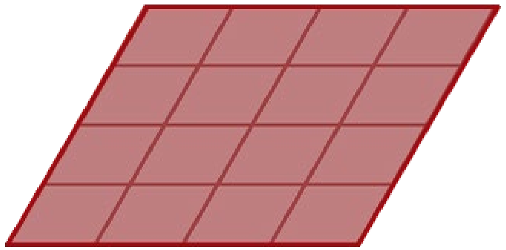
The Original RS Papers: [arXiv:hep-ph/9905221 and 9906064]

Building RS1

Previous Slide

N/A

Flat 4D Spacetime



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Begin with **flat 4D spacetime**.

Building RS1

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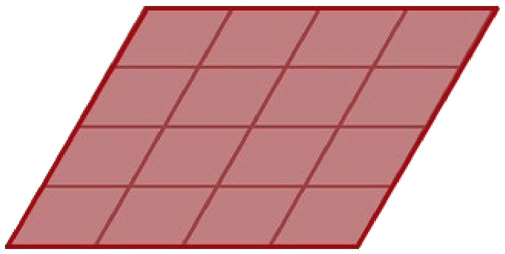
Flat 4D Spacetime



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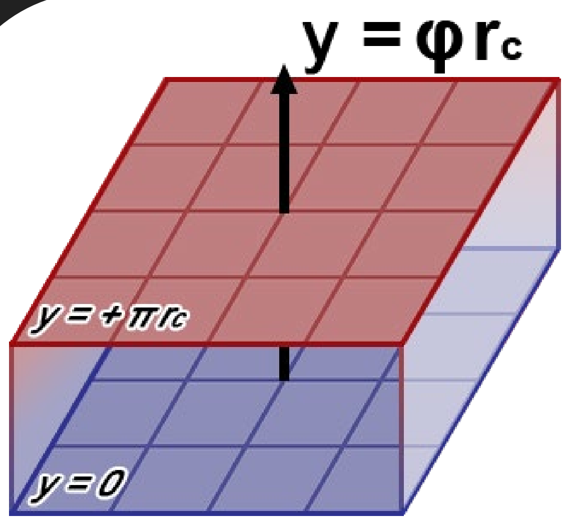
Add a Finite Extra Dimension



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$$ds^2 = dt^2 - d\vec{x}^2 - dy^2$$

Add an extra spatial dimension with coordinate **y** ranging from **y = 0** to **y = πr_c** where **r_c** is called the **compactification radius**.

Building RS1

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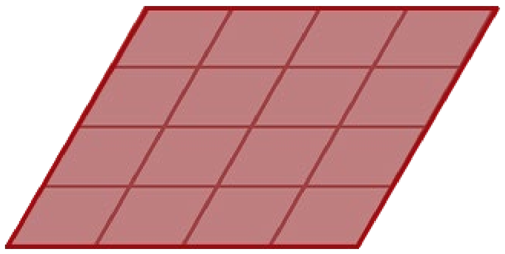
Flat 4D Spacetime



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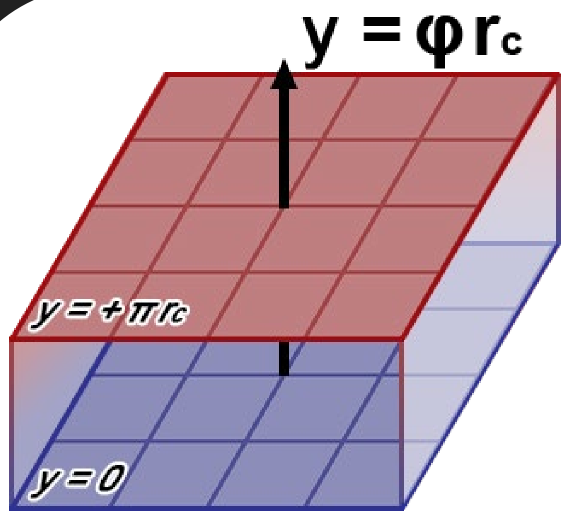

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The compactification radius $\mathbf{r_c}$ proves useful for creating dimensionless quantities. For instance, we'll sometimes replace **y** with $\varphi = y/r_c$ **∈ [0,π]**

Building RS1

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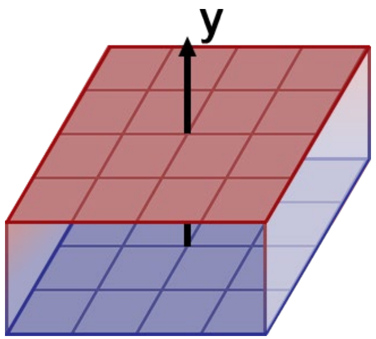
Add Extra Dimension

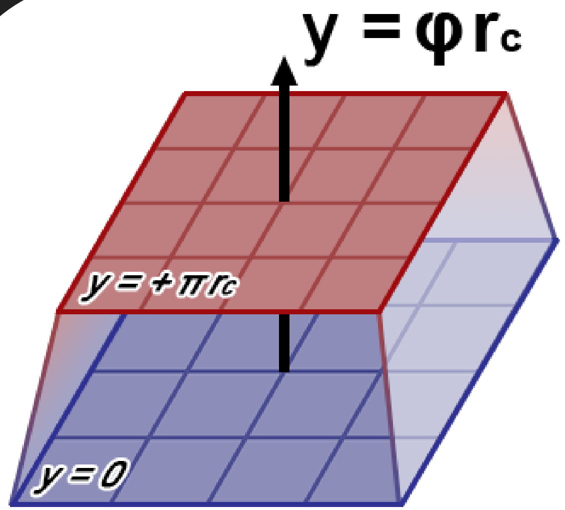


$$\eta_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

where $x^M = (x^\mu, y)$

$$\varphi \equiv \frac{y}{r_c} \in [0, \pi]$$

Warp Spacetime



$$\eta^{(\mathrm{RS})}_{MN} = \begin{pmatrix} \boxed{\varepsilon^{-2}\,\eta_{\mu\nu}} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\boxed{\varepsilon \equiv e^{+kr_c|\varphi|}}$

The **warping parameter k** (dimensions of energy) measures spacetime distortion.

The case **k = 0** describes a flat extra dimension.

Building RS1

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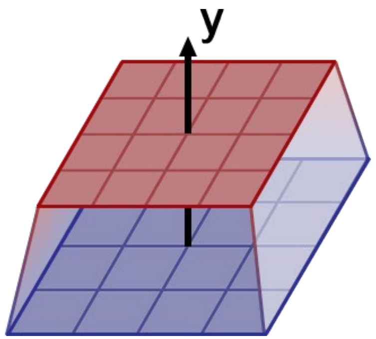
Warp Spacetime



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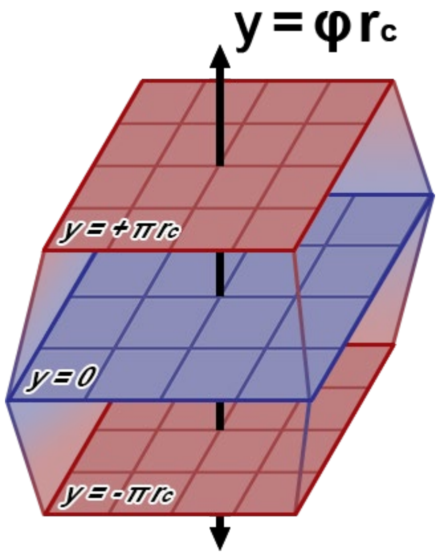
Orbifold



Extend spacetime by reflecting across **φ = 0** so that **φ ∈ [-π,+π]**.

By demanding that the interval **ds²** is invariant under this reflection, we construct an **orbifolded** theory.

This projects onto **orbifold even states** (including the **graviton** + **radion**) and eliminates orbifold odd states (graviphoton)

Building RS1

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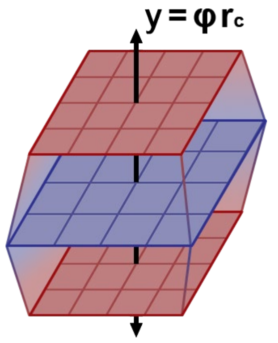
Orbifold



$$\eta^{(RS)}_{MN} = \begin{pmatrix} \varepsilon^{-2}\,\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\varepsilon \equiv e^{+kr_c|\varphi|}$

$$\varphi \equiv \frac{y}{r_c} \in [-\pi, +\pi]$$

The RS1 Metric & Field Content

$$\eta^{(RS)}_{MN} \xrightarrow{\text{perturb}} G^{(RS)}_{MN} = \begin{pmatrix} \varepsilon^{-2}\,\underline{g_{\mu\nu}}\,e^{-2\underline{\hat{u}}} & 0 \\ 0 & -(1+2\underline{\hat{u}})^2 \end{pmatrix}$$

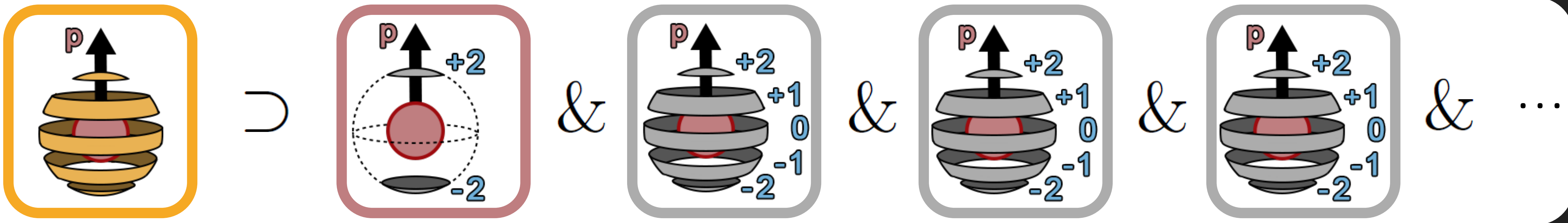
where

$$\boxed{g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa \underline{\hat{h}_{\mu\nu}}} \quad \text{and} \quad \boxed{\hat{u} \equiv \frac{\kappa}{2\sqrt{6}}\underline{\left[\varepsilon^{+2}\,e^{-\pi k r_c}\,\hat{r}\right]}}$$

$$\underline{\hat{h}_{\mu\nu}(x,y)} = \text{massless 5D spin-2}$$

$$\underline{\hat{r}(x)} = \text{massless 5D spin-0} \quad \underline{\text{(note it's } y\text{-independent!)}}$$

Building RS1: Kaluza-Klein (KK) Decomposition



$$\boxed{\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}_{\mu\nu}^{(n)}(x)\, \psi_n(\varphi)}$$

= (**4D graviton**) + (**massive 4D spin-2 KK modes**)

$$\boxed{\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}}\, \hat{r}^{(0)}(x)\, \psi_0}$$

= (**radion**)

- The **wavefunctions ψ_n** solve a specific **Sturm-Liouville Equation (S-L Eq.)**
- The solution set **{ ψ_n }** and spectrum **{ $m_n r_c$ }** are **entirely determined by kr_c.**

Building RS1: The Lagrangian

RS1 Metric

$$G_{MN}^{(\mathrm{RS})} = \begin{pmatrix} \varepsilon^{-2}\, g_{\mu\nu}\, e^{-2\hat{u}} & 0 \\ 0 & -(1+2\hat{u})^2 \end{pmatrix}$$

$$\text{where} \quad \varepsilon \equiv e^{+kr_c|\varphi|}$$

into...

$$\mathcal{L}_{\mathrm{5D}} = \underline{\mathcal{L}_{\mathrm{EH}}} + \underline{\mathcal{L}_{\mathrm{CC}}} + \underline{\Delta\mathcal{L}}$$

- **Einstein-Hilbert Lagrangian**

$$\boxed{\mathcal{L}_{\mathrm{EH}} = \frac{2}{\kappa^2}\sqrt{\det G}\; R_{\mathrm{5D}}}$$

- **Cosmological Constant Terms**

$$\boxed{\mathcal{L}_{\mathrm{CC}} = \frac{12}{\kappa^2} k r_c \left\{ 2\sqrt{\det G}\,(\partial_\varphi|\varphi|)^2 - \varepsilon^{-4}\, e^{-4\hat{u}}\,\sqrt{-\det g}\,(\partial_\varphi^2|\varphi|) \right\}}$$

to ensure 4D effective theory is flat.

- **Total Derivative**

$$\boxed{\Delta\mathcal{L} = \frac{2}{\kappa^2}\partial_y \left[\frac{\varepsilon^{-4}\, e^{-4\hat{u}}}{\sqrt{1+2\hat{u}}} \sqrt{-\det g} \left(g_{\mu\nu}^{-1}(\partial_y g^{\mu\nu}) + \partial_y \ln\left[\varepsilon^{-4}\, e^{-4\hat{u}}\right] \right) \right]}$$

to eliminate all instances of ∂_y^2 for later convenience.

Reparameterize: (k , r_c , κ) → (kr_c , m_1 , M_{Pl})

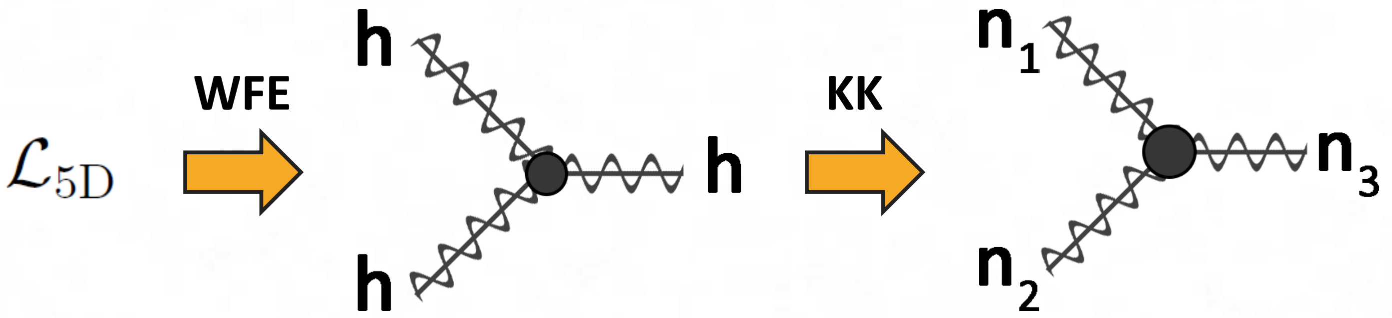
$$\boxed{\begin{aligned} kr_c &= k \cdot r_c \\ m_1 &\equiv \frac{1}{r_c}\left[(m_1 r_c)\big|_{kr_c}\right] \quad \text{via S-L Eq.} \\ M_{\text{Pl}} &\equiv \frac{2}{\kappa\sqrt{k}}\sqrt{1-e^{-2kr_c\pi}} \end{aligned}}$$

For the remainder of this presentation, whenever I show *numerical* plots, I set:

$$M_{\text{Pl}} = 2.435 \times 10^{15}\ \text{TeV}$$
$$m_1 = 1\ \text{TeV}$$

$$\mathcal{L}_{5\text{D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} + \Delta\mathcal{L} \xrightarrow[\text{\& KK}]{\text{WFE}} \mathcal{L}_{4\text{D}}^{(\text{eff})} \equiv \int dy \;\; \mathcal{L}_{5\text{D}}$$

Example: Cubic KK Mode Interaction



If **two spatial** derivatives...

If **two extra-dimensional** derivatives...

A-Type Coupling

$$\varepsilon^{-2}\ \hat{h}\,(\partial_{\mu_1}\hat{h})\,(\partial_{\mu_2}\hat{h}) \quad\longrightarrow\quad \varepsilon^{-2}\ \psi_{n_1}\,\psi_{n_2}\,\psi_{n_3}$$

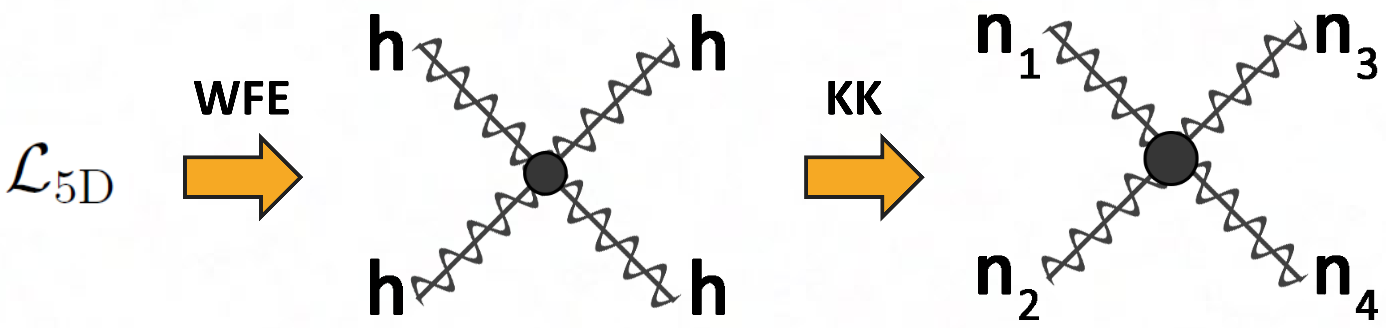
$$\longrightarrow\ \boxed{a_{n_1 n_2 n_3} \equiv \frac{1}{\pi r_c}\int d\varphi \quad \varepsilon^{-2}\ \psi_{n_1}\,\psi_{n_2}\,\psi_{n_3}}$$

B-Type Coupling

$$\varepsilon^{-4}\ \hat{h}\,(\partial_{\varphi}\hat{h})\,(\partial_{\varphi}\hat{h}) \quad\longrightarrow\quad \varepsilon^{-4}\ (\partial_{\varphi}\psi_{n_1})\,(\partial_{\varphi}\psi_{n_2})\,\psi_{n_3}$$

$$\longrightarrow\ \boxed{b_{n_1 n_2 n_3} \equiv \frac{1}{\pi r_c}\int d\varphi \quad \varepsilon^{-4}\ (\partial_{\varphi}\psi_{n_1})\,(\partial_{\varphi}\psi_{n_2})\,\psi_{n_3}}$$

Example: Quartic KK Mode Interaction



If **two spatial** derivatives...

A-Type Coupling

$$\varepsilon^{-2}\ \hat{h}\,\hat{h}\,(\partial_{\mu_1}\hat{h})\,(\partial_{\mu_2}\hat{h}) \ \longrightarrow\ \varepsilon^{-2}\ \psi_{n_1}\,\psi_{n_2}\,\psi_{n_3}\,\psi_{n_4}$$

$$\boxed{a_{n_1n_2n_3n_4} \equiv \frac{1}{\pi r_c}\int d\varphi \quad \varepsilon^{-2}\ \psi_{n_1}\,\psi_{n_2}\,\psi_{n_3}\,\psi_{n_3}}$$

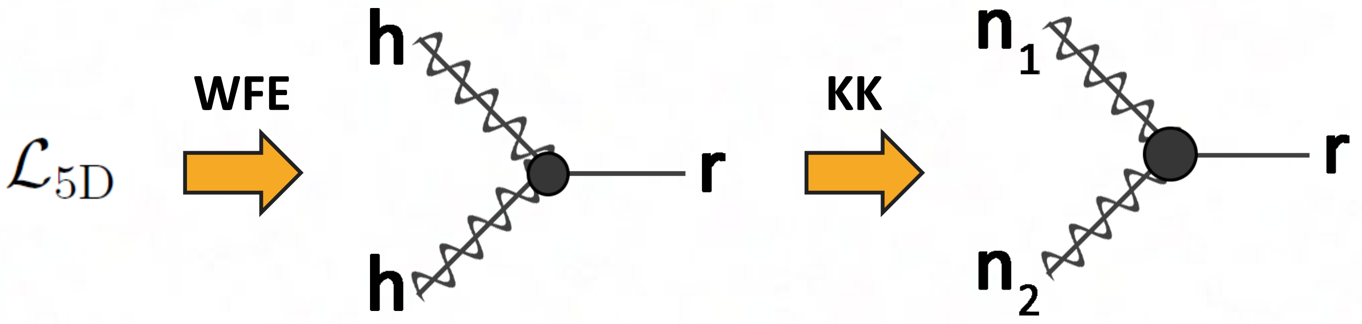
If **two extra-dimensional** derivatives...

B-Type Coupling

$$\varepsilon^{-4}\ \hat{h}\,\hat{h}\,(\partial_{\varphi}\hat{h})\,(\partial_{\varphi}\hat{h}) \ \longrightarrow\ \varepsilon^{-4}\ (\partial_{\varphi}\psi_{n_1})\,(\partial_{\varphi}\psi_{n_2})\,\psi_{n_3}\,\psi_{n_4}$$

$$\boxed{b_{n_1n_2n_3n_3} \equiv \frac{1}{\pi r_c}\int d\varphi \quad \varepsilon^{-4}\ (\partial_{\varphi}\psi_{n_1})\,(\partial_{\varphi}\psi_{n_2})\,\psi_{n_3}\,\psi_{n_4}}$$

Example: (KK Mode)^2 - Radion Interaction



If **two extra-dimensional** derivatives...

B-Type Coupling

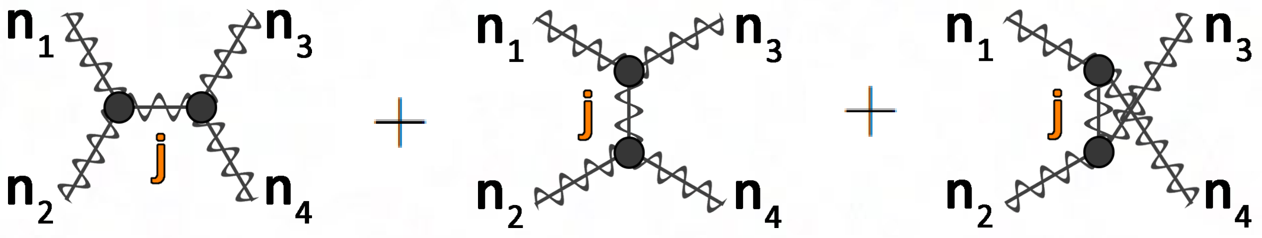
$$\varepsilon^{-4}\ \underline{\left[\varepsilon^{+2}\, e^{-\pi k r_c}\, \hat{r}\right]}\ (\partial_\varphi \hat{h})\, (\partial_\varphi \hat{h}) \ \longrightarrow\ \varepsilon^{-2}\ (\partial_\varphi \psi_{n_1})\, (\partial_\varphi \psi_{n_2})\, \psi_0$$

$$\longrightarrow\ \boxed{b_{n_1 n_2 r} \equiv \frac{\psi_0}{\pi r_c}\, e^{-\pi k r_c} \int d\varphi\ \ \varepsilon^{-2}\ (\partial_\varphi \psi_{n_1})\, (\partial_\varphi \psi_{n_2})}$$

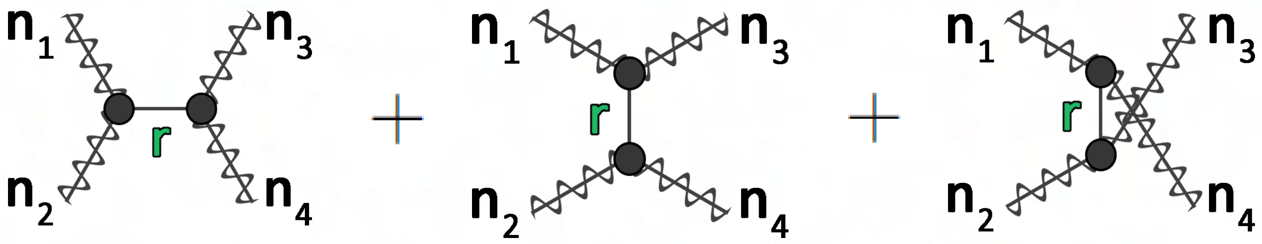
No A-Type (KK Mode)^2 - Radion Interaction Appears in L_{5D}

Matrix Element: Total Matrix Element

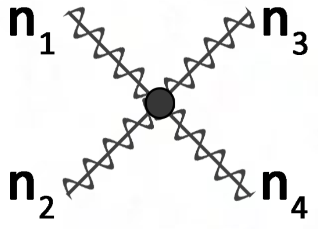
$\mathcal{M}_j =$



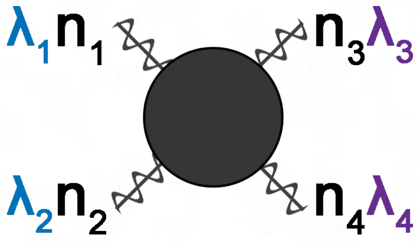
$\mathcal{M}_r =$



$\mathcal{M}_c =$

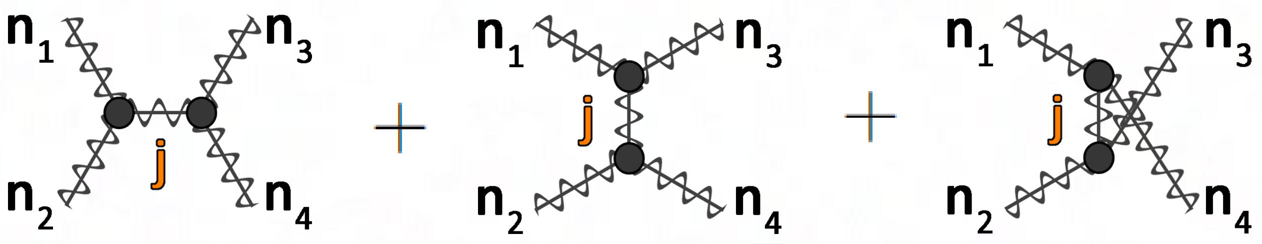


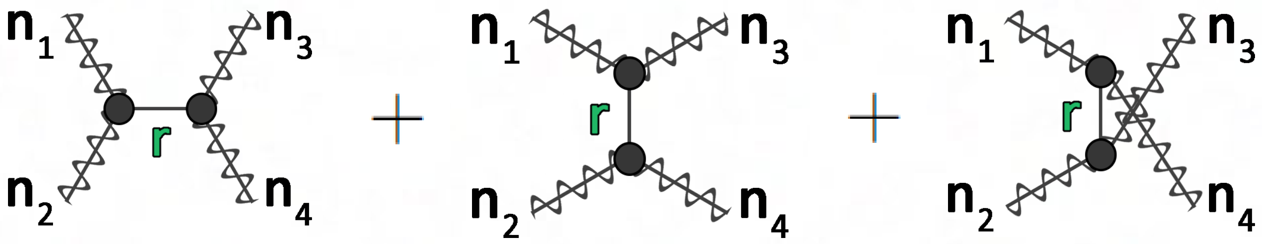
$\mathcal{M}_{\text{full}} =$

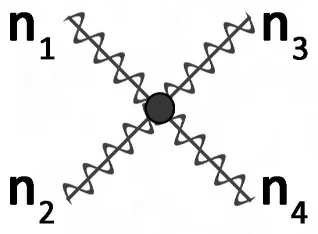


$$= \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{+\infty} \mathcal{M}_j$$

Matrix Element: Truncated Matrix Element

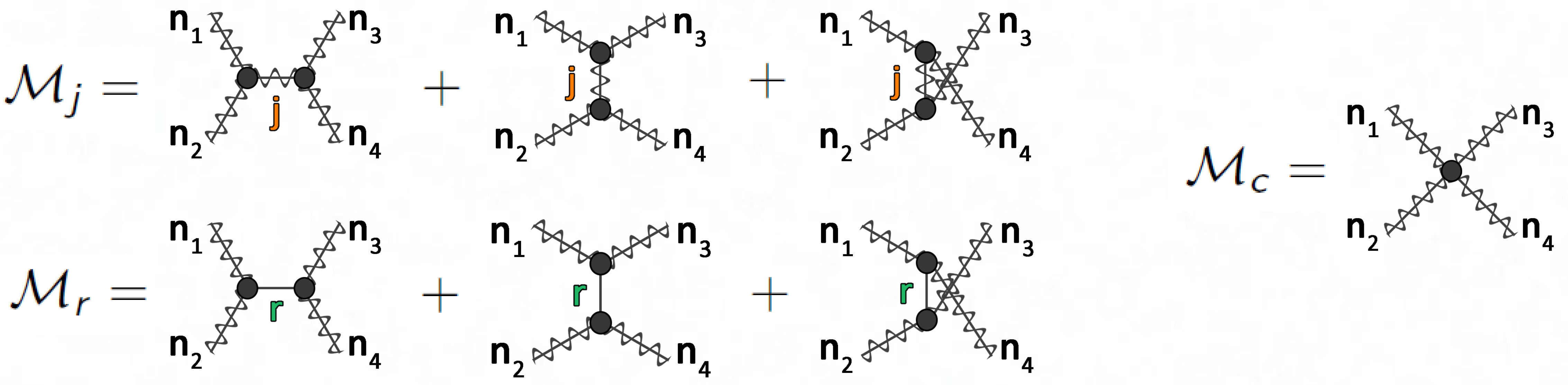
$\mathcal{M}_j =$ 

$\mathcal{M}_r =$ 

$\mathcal{M}_c =$ 

$$\mathcal{M}^{[\underline{N}]} \equiv \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{\underline{N}} \mathcal{M}_j$$

Matrix Element: Expanded Matrix Element



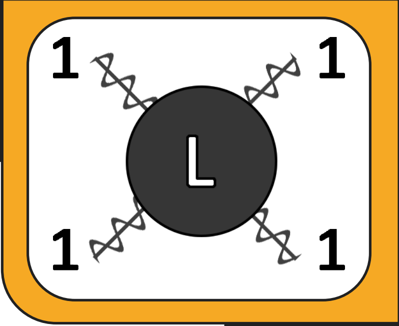
$$\mathcal{M}^{[N]} \equiv \sum_{\sigma} \underline{\mathcal{M}^{[N](\sigma)}} \equiv \sum_{\sigma} \underline{\overline{\mathcal{M}}^{[N](\sigma)}} \, (E r_c)^{\underline{2\sigma}}$$

Expectation:
$M^{[N](\sigma)}$ vanishes as $N \rightarrow \infty$ for all $\sigma > 1$ (i.e. growth > E^2)

Cancelling Bad High-Energy Growth

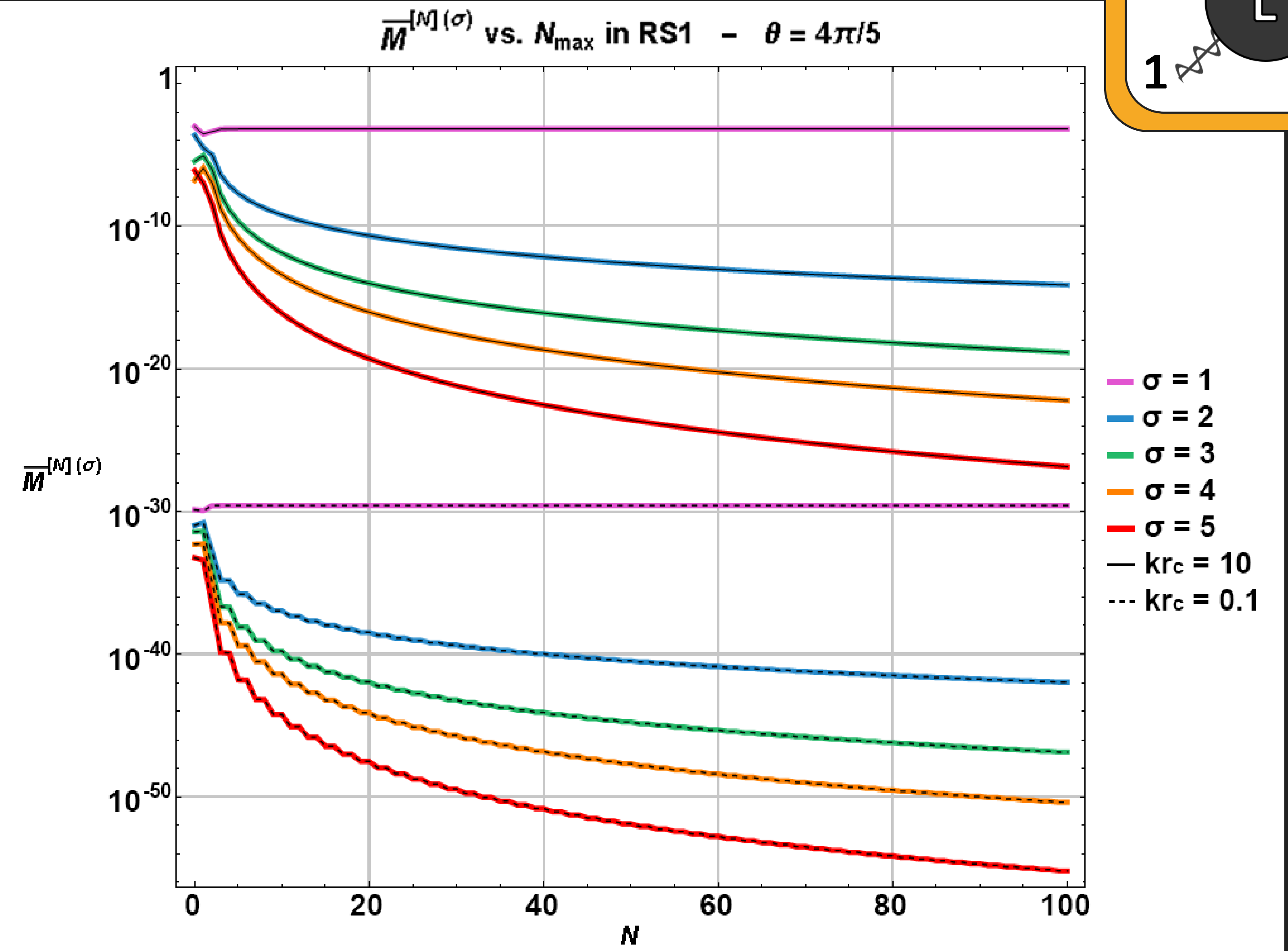
Sum Rules & Numerical Checks

Longitudinal (1,1) → (1,1) : Cancellations



$$\mathcal{M}^{[N]} = \sum_{\sigma} \overline{\mathcal{M}}^{[N](\sigma)} \, (E r_c)^{2\sigma}$$

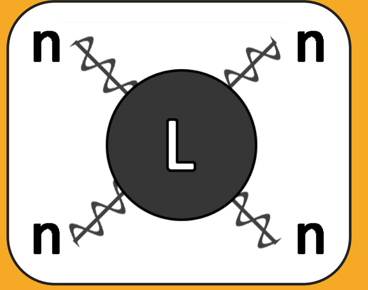
$$\mathcal{M}^{[N]} \equiv \text{(1st } N \text{ KK Modes)}$$



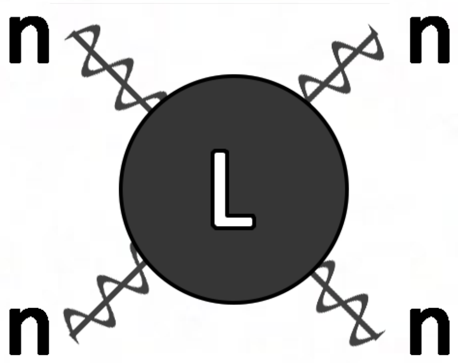
Q: How does including more terms in the truncated sum affect leading energy growth?

A: By increasing N, any energy growth faster than O(E^2) increasingly cancels. O(E^2) remains.

Rewrite Generic Elastic Couplings



How does this happen analytically?

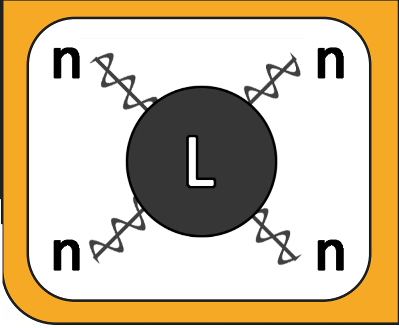


Focus on the **longitudinal elastic scattering matrix elements**, which seemingly grow like **$O(E^{10})$** at finite truncation.

Use wavefxn properties to rewrite **B-Type couplings** as **A-Type couplings**:

$$b_{nnj} = \left[(m_n r_c)^2 - \tfrac{1}{2}(m_j r_c)^2\right] a_{nnj}$$
$$b_{jnn} = \tfrac{1}{2}(m_j r_c)^2 a_{nnj}$$
$$b_{nnnn} = \tfrac{1}{3}(m_n r_c)^2 a_{nnnn}$$

Sum Rules (order-by-order)



The **longitudinal elastic scattering matrix elements** grows like O(E^2) iff the following **sum rules** between masses & couplings hold true:

$$\underline{\mathcal{O}(s^5)}: \qquad \sum_{j=0}^{+\infty} a_{nnj}^2 = a_{nnnn}$$

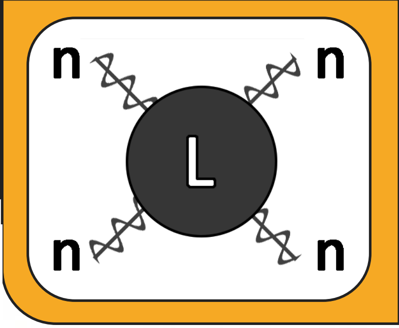
$$\underline{\mathcal{O}(s^4)}: \qquad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^2 a_{nnj}^2 = \frac{4}{3}\, a_{nnnn}$$

Reorganize These Last 2

$$\underline{\mathcal{O}(s^3)}: \qquad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = \frac{4}{5}\left[9\,\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right] + \frac{16}{15}\, a_{nnnn}$$

$$\underline{\mathcal{O}(s^2)}: \qquad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^6 a_{nnj}^2 = 4\left[9\,\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]$$

Sum Rules (alternate set)



The **longitudinal elastic scattering matrix elements** grows like O(E^2) iff the following **sum rules** between masses & couplings hold true:

$$\underline{\mathcal{O}(s^5)} : \quad \sum_{j=0}^{+\infty} a_{nnj}^2 = a_{nnnn}$$ **PROVED**

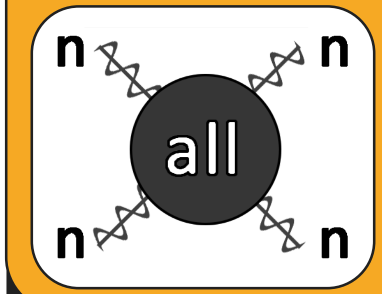
$$\underline{\mathcal{O}(s^4)} : \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^2 a_{nnj}^2 = \frac{4}{3} a_{nnnn}$$ **PROVED**

$$\underline{\mathcal{O}(s^2) \leftrightarrow \mathcal{O}(s^3)} : \quad \sum_{j=0}^{+\infty} \left[\left(\frac{m_j}{m_n}\right)^2 - 5\right] \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = -\frac{16}{3} a_{nnnn}$$ **PROVED**

$$\underline{\mathcal{O}(s^2)} : \quad \boxed{\sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^6 a_{nnj}^2 = 4\left[9\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]}$$

This is the only rule left without an analytic proof.

Elastic 2-to-2 KK Mode Scattering Matrix Elements in RS1
Fastest Energy Growth per Helicity Combination: $(\lambda_1,\lambda_2) \to (\lambda_3,\lambda_4)$



BEFORE SUM RULES

λ_1	λ_3 \ λ_2, λ_4	-2, -2	-2, -1	-2, 0	-2, +1	-2, +2	-1, -2	-1, -1	-1, 0	-1, +1	-1, +2	0, -2	0, -1	0, 0	0, +1	0, +2	+1, -2	+1, -1	+1, 0	+1, +1	+1, +2	+2, -2	+2, -1	+2, 0	+2, +1	+2, +2
-2	-2	2	3	4	3	2	3	4	5	4	3	4	5	6	5	4	3	4	5	4	3	2	3	4	3	2
	-1	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	0	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	+1	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	+2	2	3	4	3	2	3	4	5	4	3	4	5	6	5	4	3	4	5	4	3	2	3	4	3	2
-1	-2	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	-1	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	0	5	6	7	6	5	6	7	8	7	6	7	8	9	8	7	6	7	8	7	6	5	6	7	6	5
	+1	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	+2	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
0	-2	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	-1	5	6	7	6	5	6	7	8	7	6	7	8	9	8	7	6	7	8	7	6	5	6	7	6	5
	0	6	7	8	7	6	7	8	9	8	7	8	9	10	9	8	7	8	9	8	7	6	7	8	7	6
	+1	5	6	7	6	5	6	7	8	7	6	7	8	9	8	7	6	7	8	7	6	5	6	7	6	5
	+2	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
+1	-2	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	-1	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	0	5	6	7	6	5	6	7	8	7	6	7	8	9	8	7	6	7	8	7	6	5	6	7	6	5
	+1	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	+2	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
+2	-2	2	3	4	3	2	3	4	5	4	3	4	5	6	5	4	3	4	5	4	3	2	3	4	3	2
	-1	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	0	4	5	6	5	4	5	6	7	6	5	6	7	8	7	6	5	6	7	6	5	4	5	6	5	4
	+1	3	4	5	4	3	4	5	6	5	4	5	6	7	6	5	4	5	6	5	4	3	4	5	4	3
	+2	2	3	4	3	2	3	4	5	4	3	4	5	6	5	4	3	4	5	4	3	2	3	4	3	2

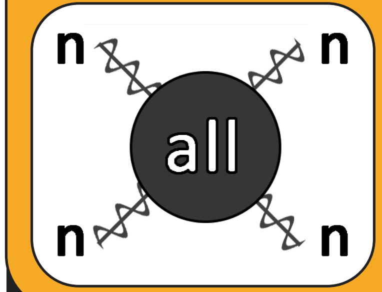
→

λ_3 \ λ_4	-2	-1	0	+1	+2
-2	6	7	8	7	6
-1	7	8	9	8	7
0	8	9	10	9	8
+1	7	8	9	8	7
+2	6	7	8	7	6

Legend

X X X = $O(E^x)$ growth

Elastic 2-to-2 KK Mode Scattering Matrix Elements in RS1
Fastest Energy Growth per Helicity Combination: $(\lambda_1,\lambda_2) \rightarrow (\lambda_3,\lambda_4)$



$\lambda_1 = -2$

λ_2 = -2					-1					0					+1					+2				
2	-1	0	-3	-2	-1	2	1	0	-1	0	1	2	1	0	-3	0	1	2	1	-2	-1	0	1	2
-1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	-1	2	1	0	-1	0	1	2	1
0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0
-3	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1
-2	-3	-4	-5	-6	-1	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2

$\lambda_1 = -1$

λ_2 = -2					-1					0					+1					+2				
-1	2	1	0	-1	0	1	2	-1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1
0	-1	-2	-3	-4	-1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	-1	0
-1	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-3

$\lambda_1 = 0$

λ_2 = -2					-1					0					+1					+2				
0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0
1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1
0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0

$\lambda_1 = +1$

λ_2 = -2					-1					0					+1					+2				
-3	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-1
0	-1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	-1	-4	-3	-2	-1	0
1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	-1	2	1	0	-1	0	1	2	-1

$\lambda_1 = +2$

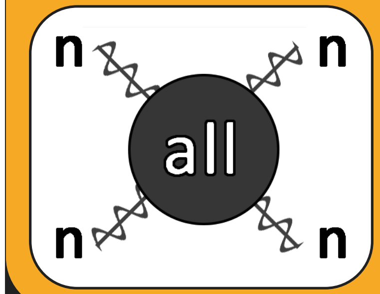
λ_2 = -2					-1					0					+1					+2				
-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-1	-6	-5	-4	-3	-2
-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-3
0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0
1	2	1	0	-1	0	1	2	-1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	-1
2	1	0	-1	-2	1	2	1	0	-3	0	1	2	1	0	-1	0	1	2	-1	-2	-3	0	-1	2

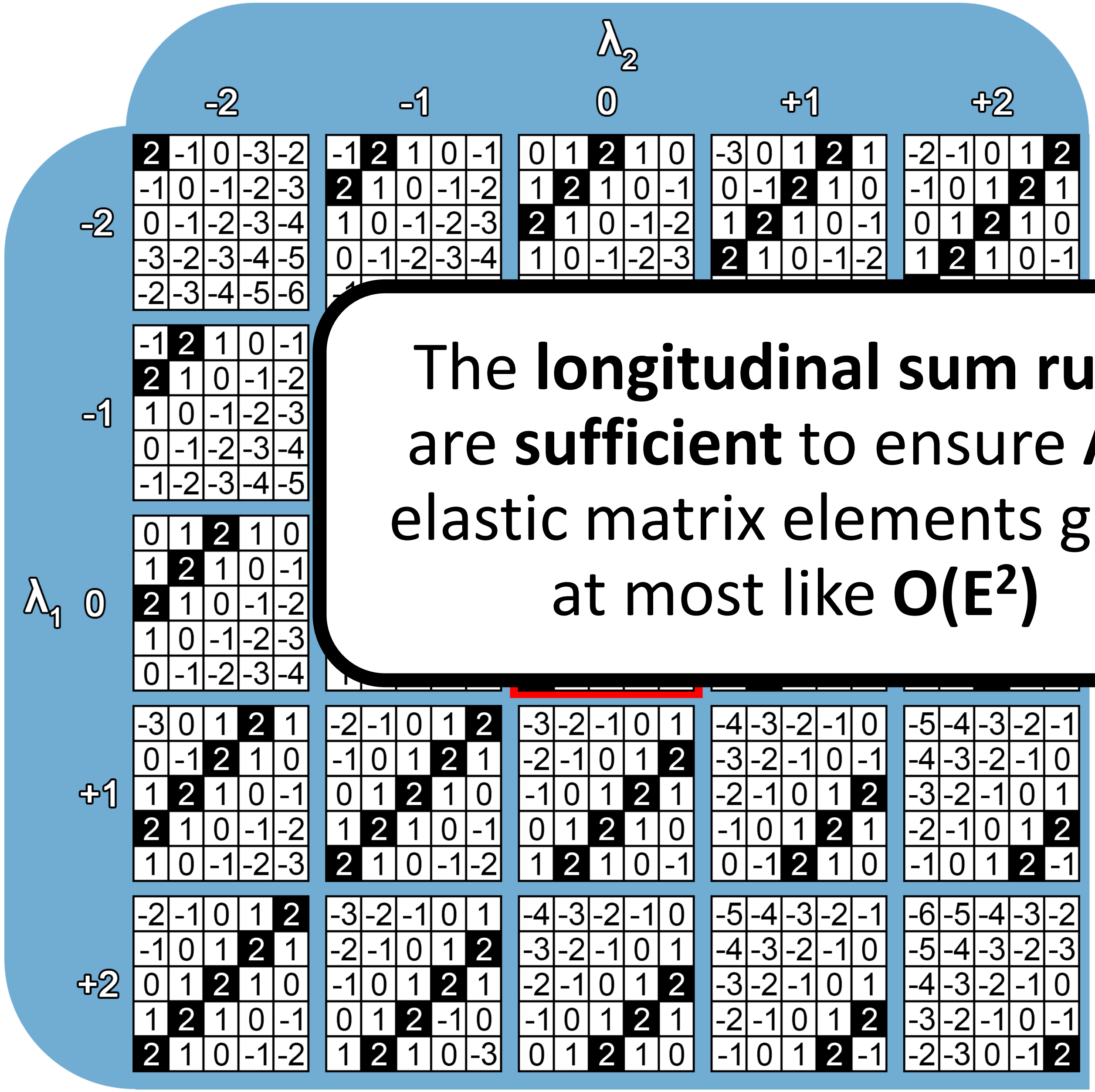
λ_3 \ λ_4	-2	-1	0	+1	+2
-2	-2	-1	0	1	2
-1	-1	0	1	2	1
0	0	1	2	1	0
+1	1	2	1	0	-1
+2	2	1	0	-1	-2

Legend

X = X = $O(E^x)$ growth

Elastic 2-to-2 KK Mode Scattering Matrix Elements in RS1
Fastest Energy Growth per Helicity Combination: $(\lambda_1,\lambda_2) \rightarrow (\lambda_3,\lambda_4)$





The **longitudinal sum rules** are **sufficient** to ensure **ALL** elastic matrix elements grow at most like **O(E²)**

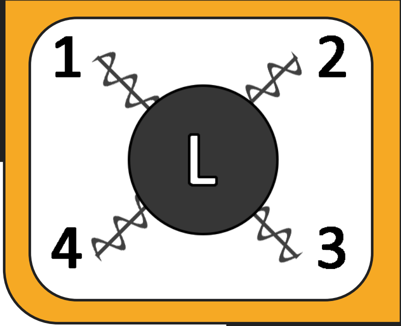
λ_4

	-2	-1	0	+1	+2
-2	-2	-1	0	1	2
-1	-1	0	1	2	1
0	0	1	2	1	0
+1	1	2	1	0	-1
+2	2	1	0	-1	-2

Legend

X = X = $O(E^X)$ growth

Longitudinal (1,4) → (2,3) : Cancellations

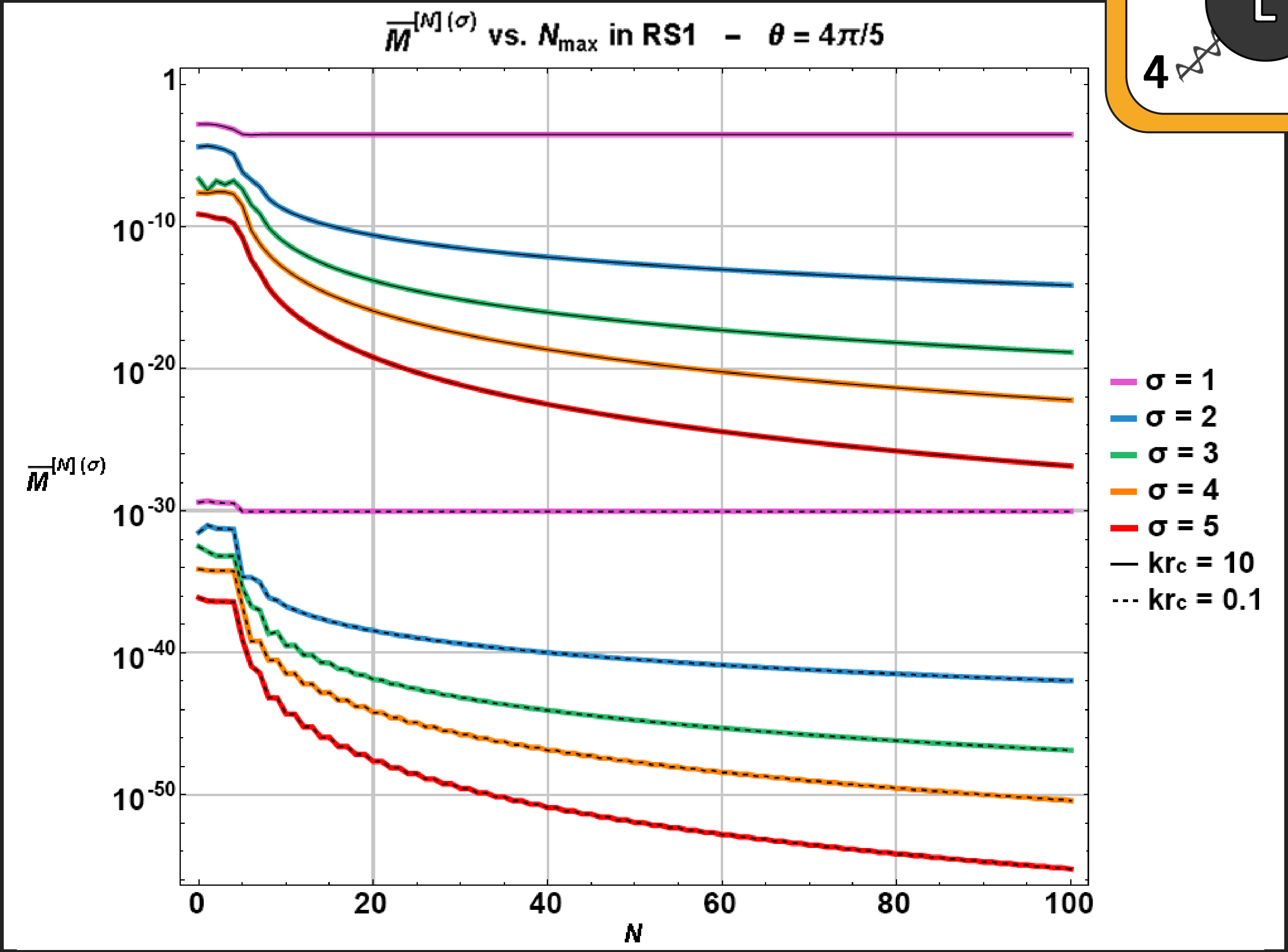


$$\mathcal{M}^{[N]} = \sum_{\sigma} \overline{\mathcal{M}}^{[N](\sigma)} \, (E r_c)^{2\sigma}$$

$$\mathcal{M}^{[N]} \equiv \text{(1st } N \text{ KK Modes)}$$

Q: How does including more terms in the truncated sum affect leading energy growth?

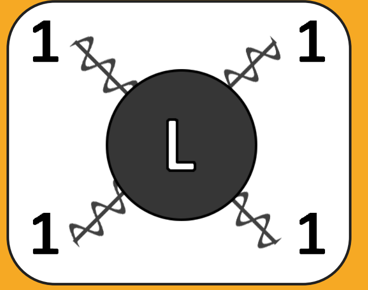
A: By increasing N, any energy growth faster than O(E^2) increasingly cancels. O(E^2) remains.

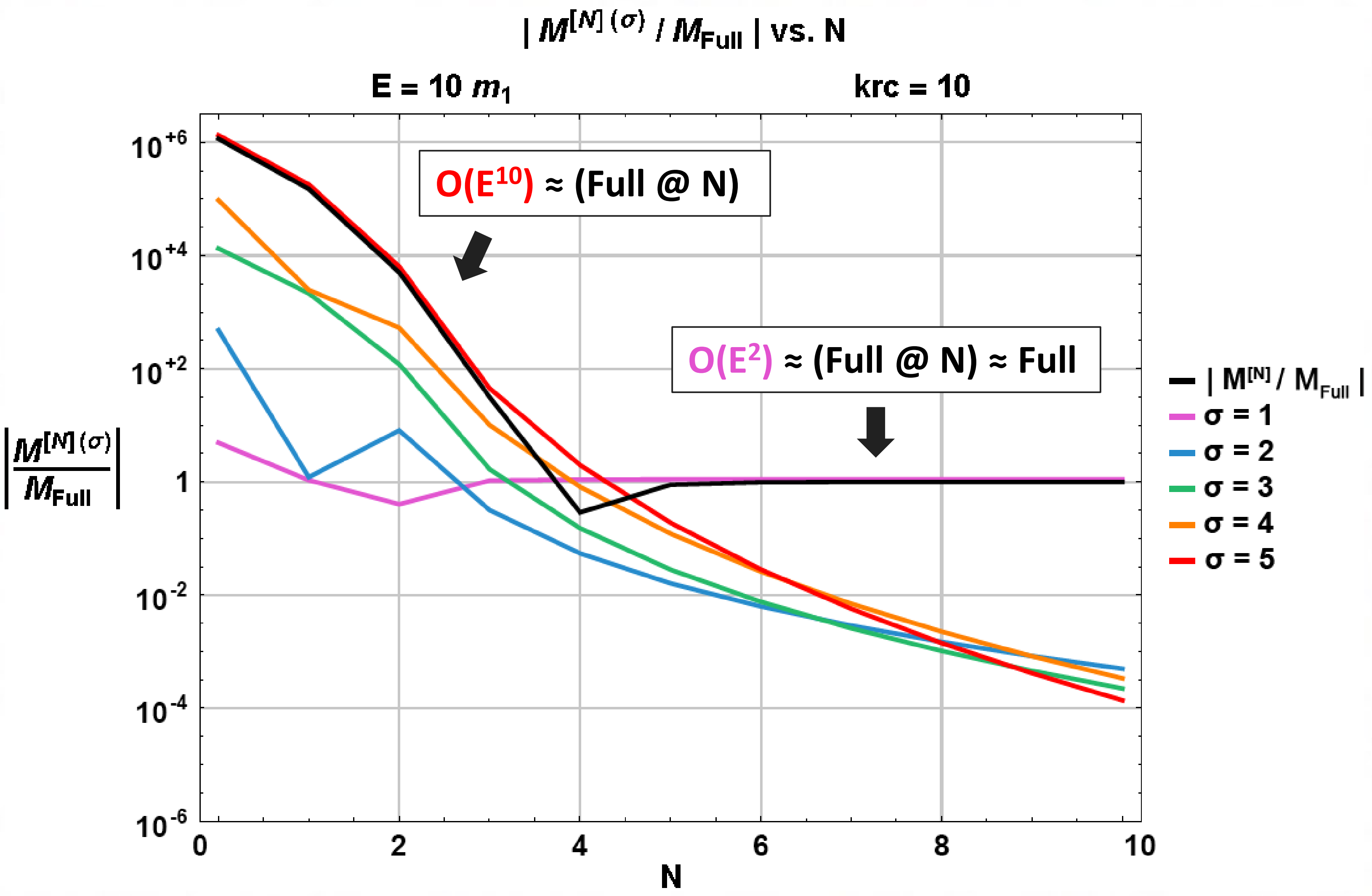


Finite Truncation at Low Energy

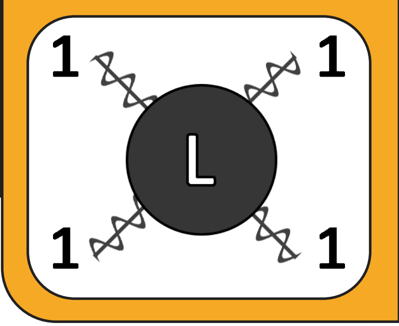
How many states should I include?

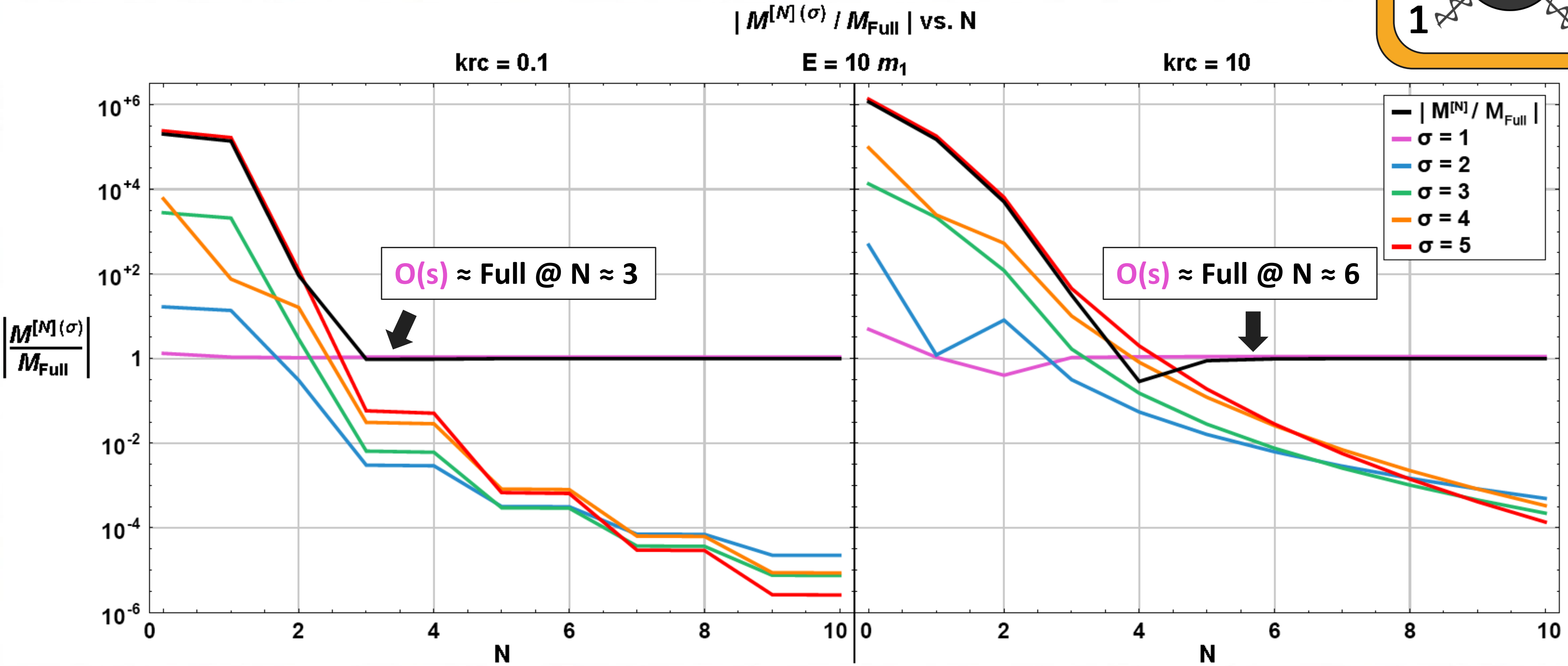
Truncation at Low Energies: $E = 10\ m_1$



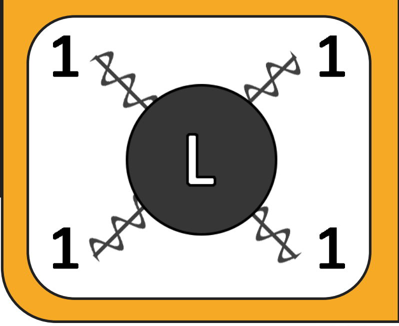


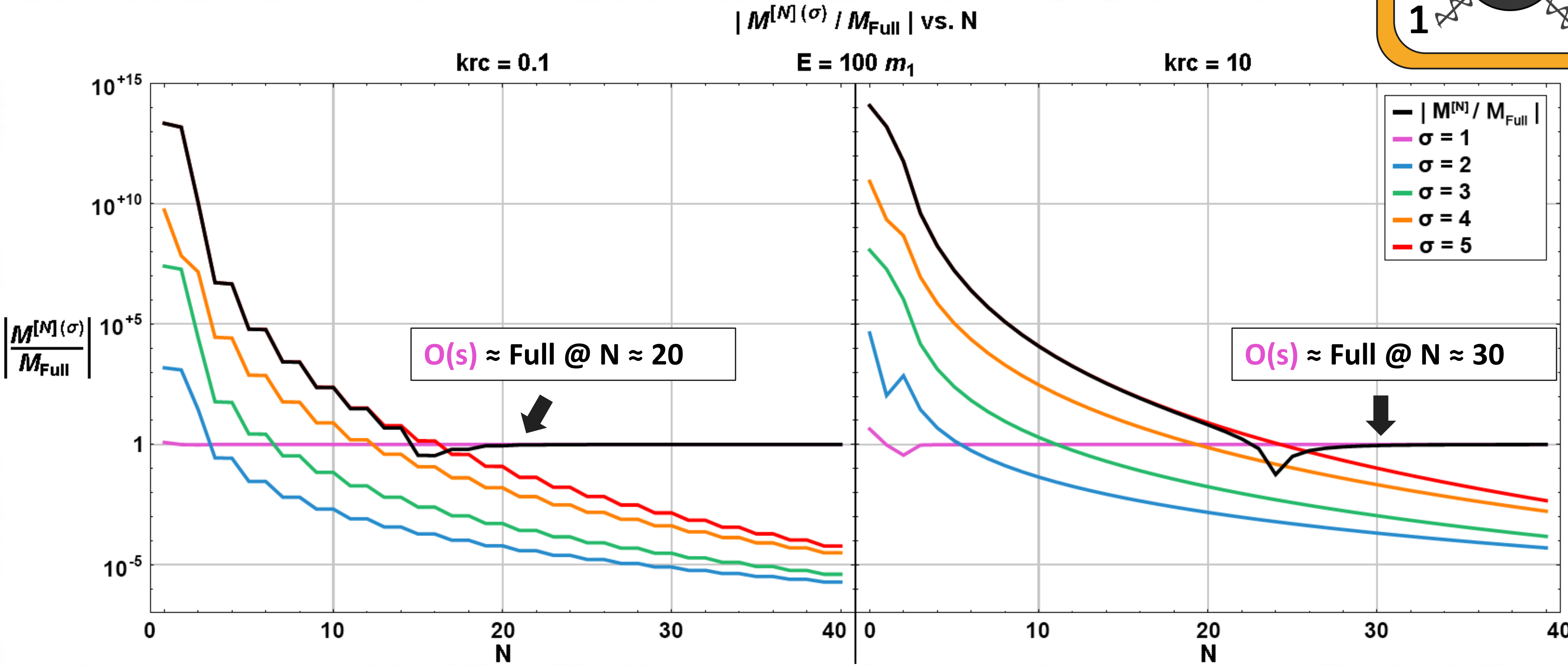
Truncation at Low Energies: $E = 10\ m_1$





Truncation at Low Energies: E = 100 m_1





Conclusion

- **<u>Interested in learning more?</u>** These results (and more!) will be available on the arXiv this month.
  - **See also:** [arXiv:1906.11098] ; [arXiv:1910.06159]
- **<u>What's next?:</u>** ● radion stabilization ● dark matter applications
  ● high-energy behavior of RS1 bulk/brane scalar matter

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Thank you for attending!