Spin-2 KK Mode Scattering in the Truncated RS1

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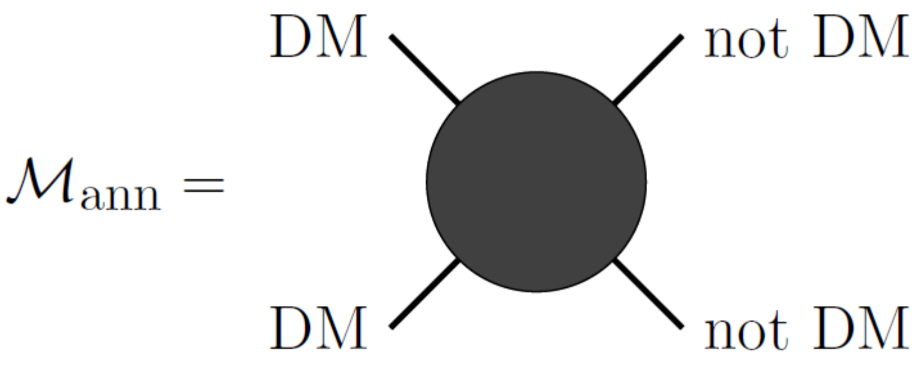
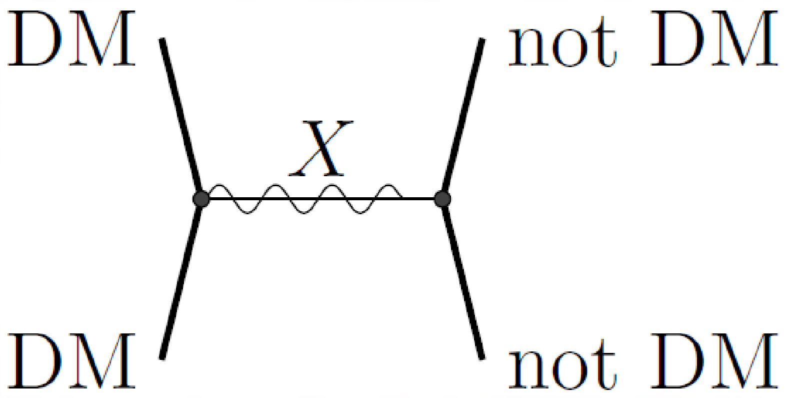
Any dark matter (DM) model should predict *at most* as much dark matter as measured.

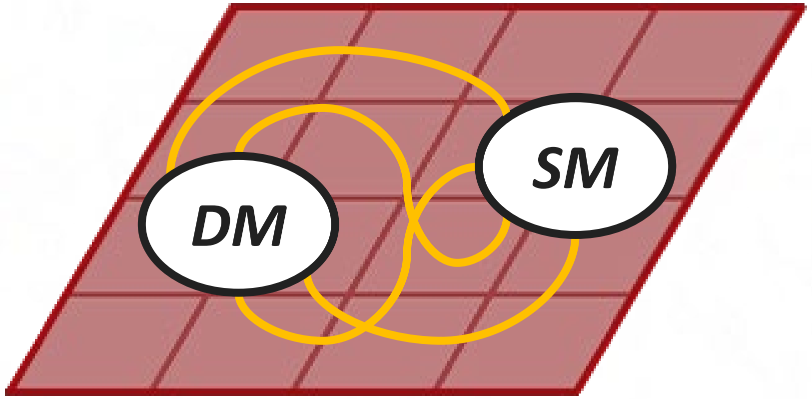
- **Freeze-Out Model:**
  DM is in thermal equilibrium until it gets too cold & disperse to annihilate anymore; particle number becomes constant.

- **Portal Models:**
  Eliminate a lot of DM quickly before freeze-out by going on resonance.





Motivating a Massive Spin-2 Portal



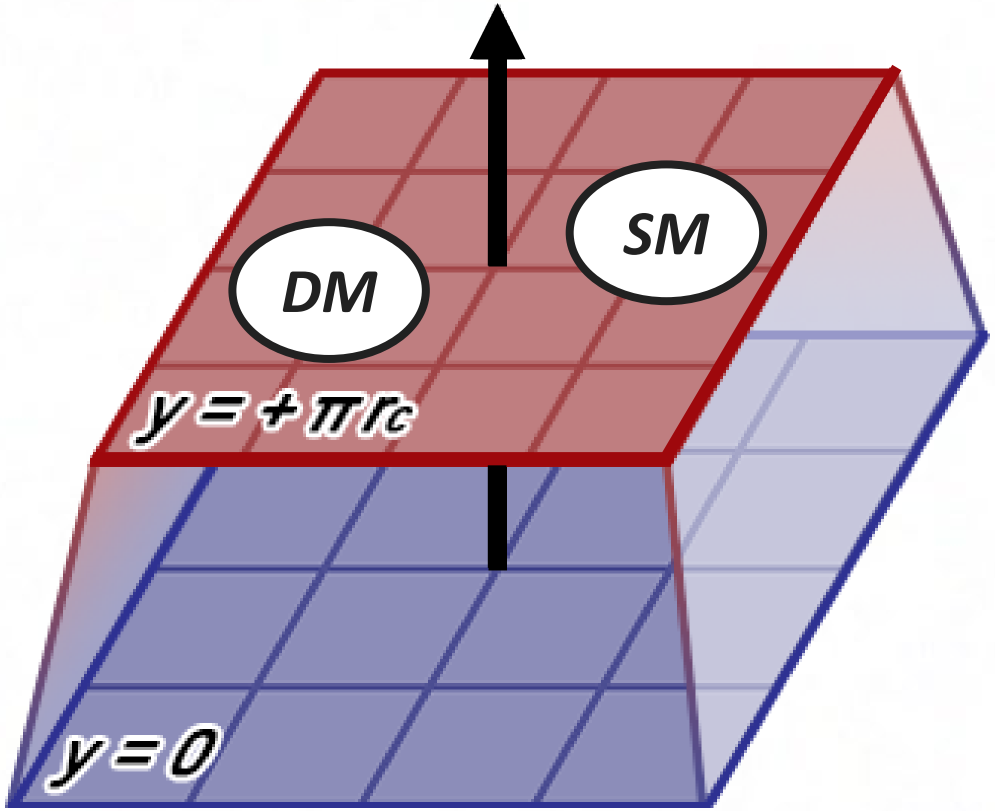
Gravity

4D Gravity

$$\Lambda_{\text{strong}} \sim M_{\text{Pl}}$$

Usual 4D Spacetime

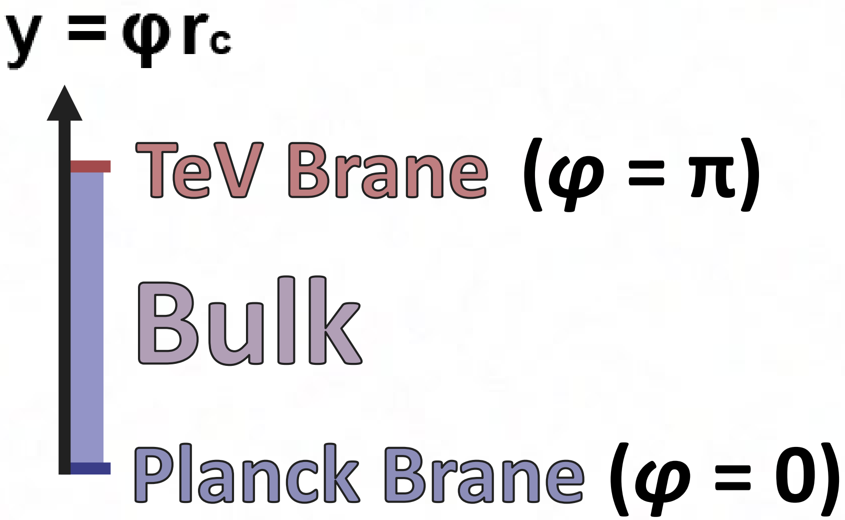
Motivating a Massive Spin-2 Portal



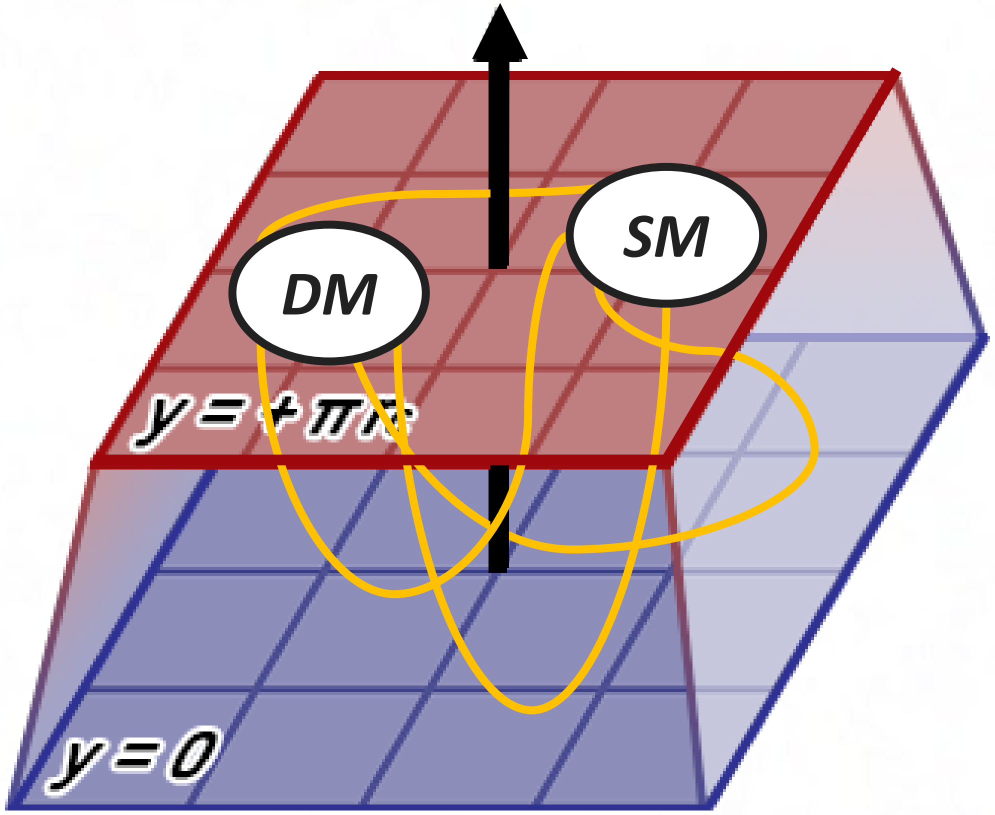
Randall-Sundrum 1

…aka a section of AdS5

…aka the model this whole talk is about

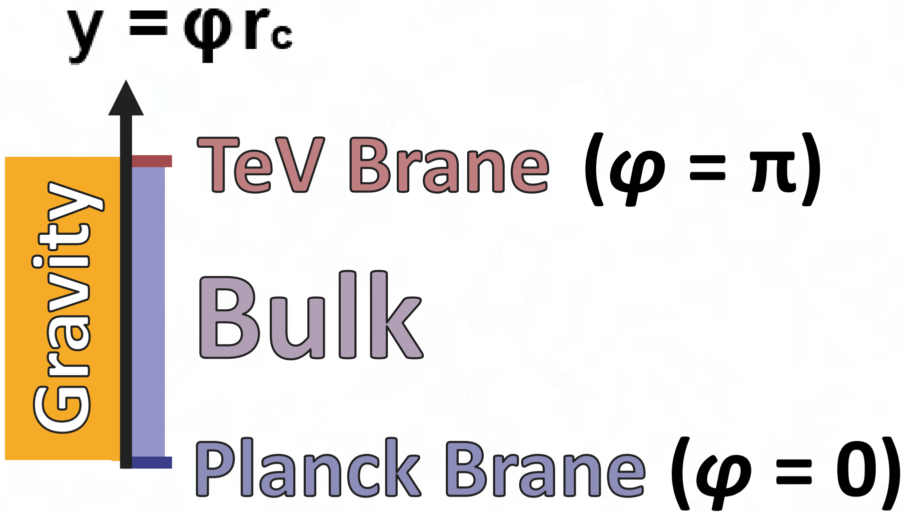


Motivating a Massive Spin-2 Portal

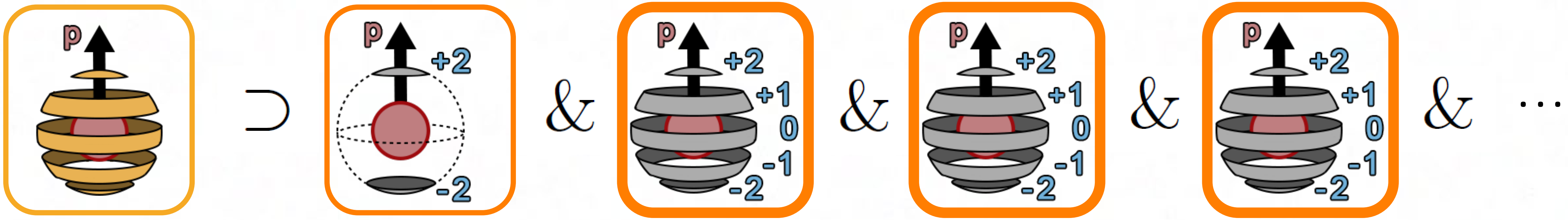


Randall-Sundrum 1

$$\Lambda_{\text{strong}} \sim \Lambda_\pi \equiv M_{\text{Pl}}\, e^{-\pi k r_c}$$



RS1 → Many Massive Spin-2 Portals

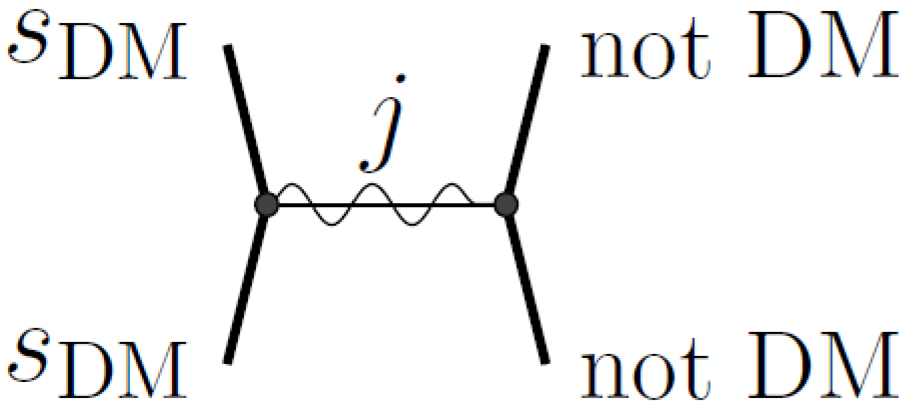


$$m_0 = 0 < m_1 < m_2 < m_3 < \ldots$$

(I'll talk A LOT more about this mode decomposition later.)

Countably infinite portals!

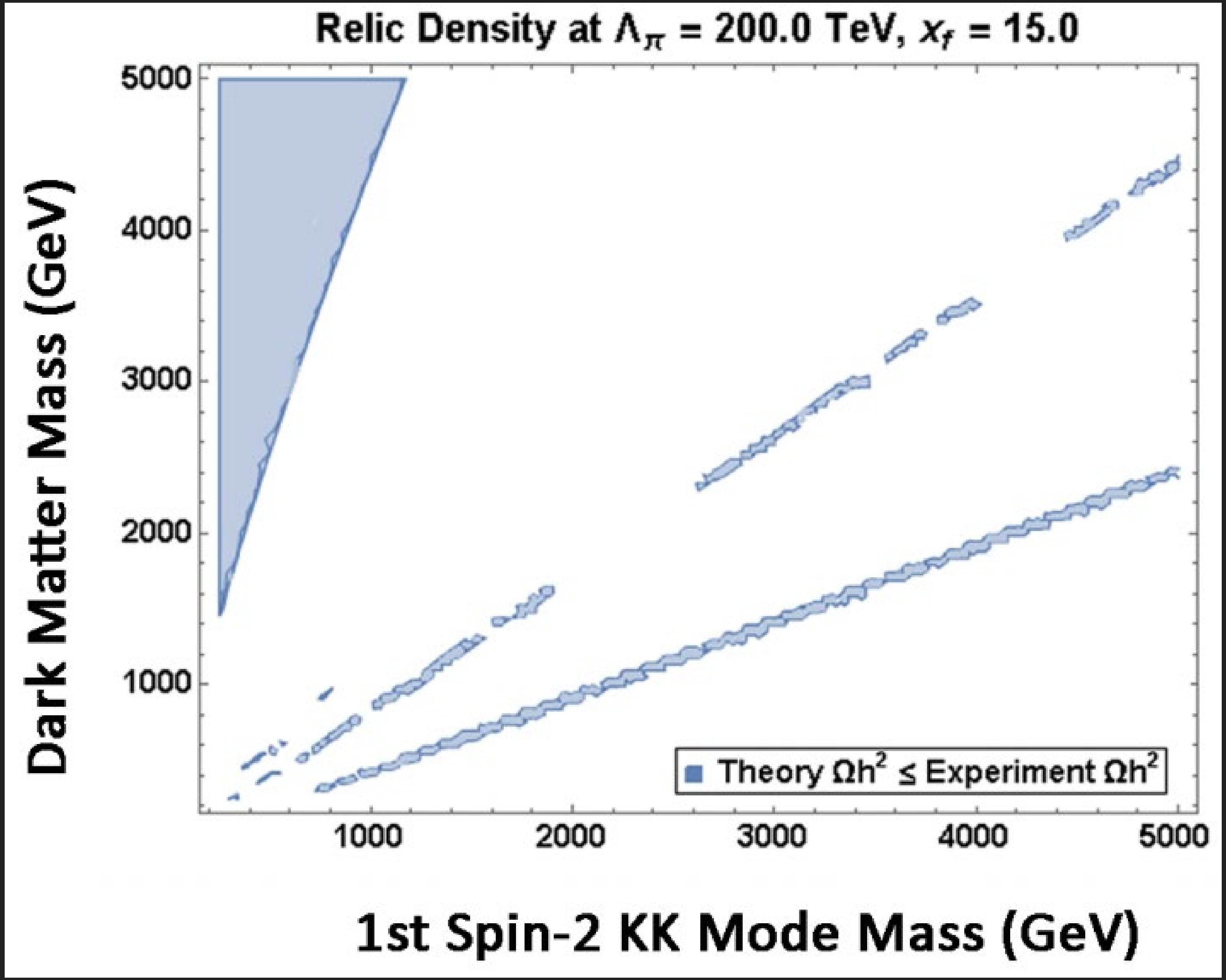
$$\sum_{j=0}^{+\infty}$$



<u>Practical:</u>
truncate to
1st ~five states

A Mysterious Region?

$$\Lambda_\pi \equiv M_{\rm Pl}\, e^{-\pi k r_c}$$

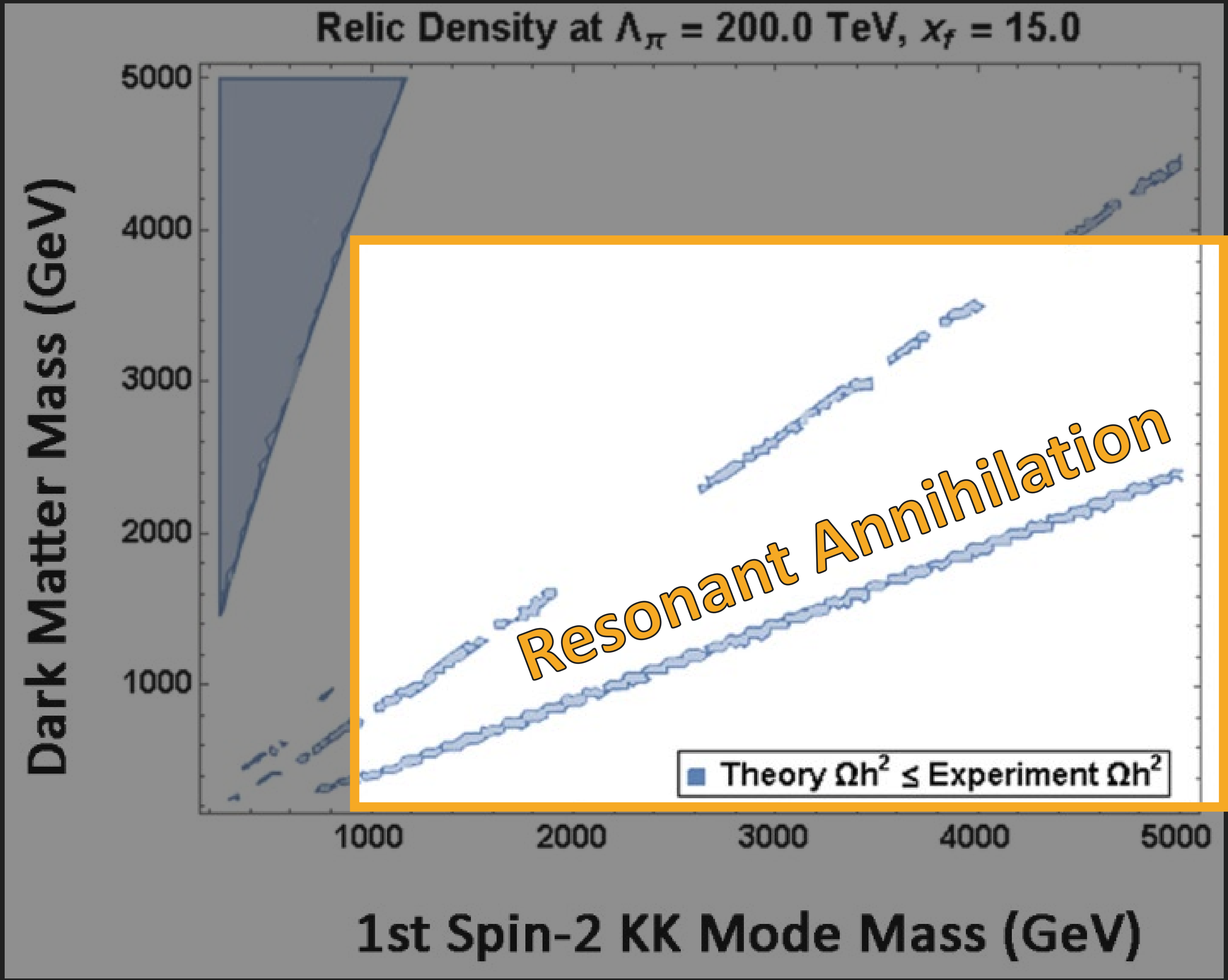


Key

Allowed

Dark Matter Overabundant

A Mysterious Region? $\Lambda_\pi \equiv M_{\rm Pl}\, e^{-\pi k r_c}$



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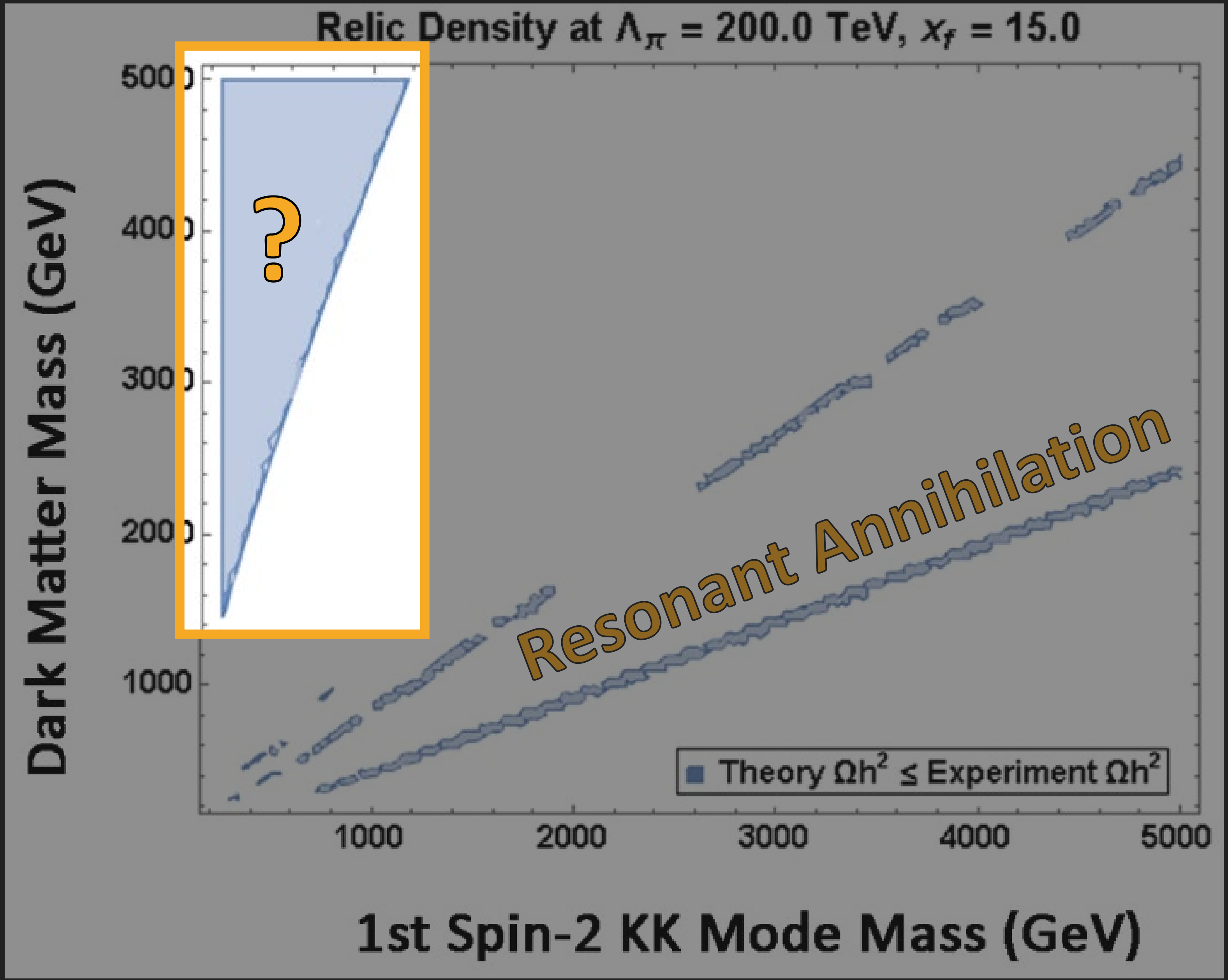

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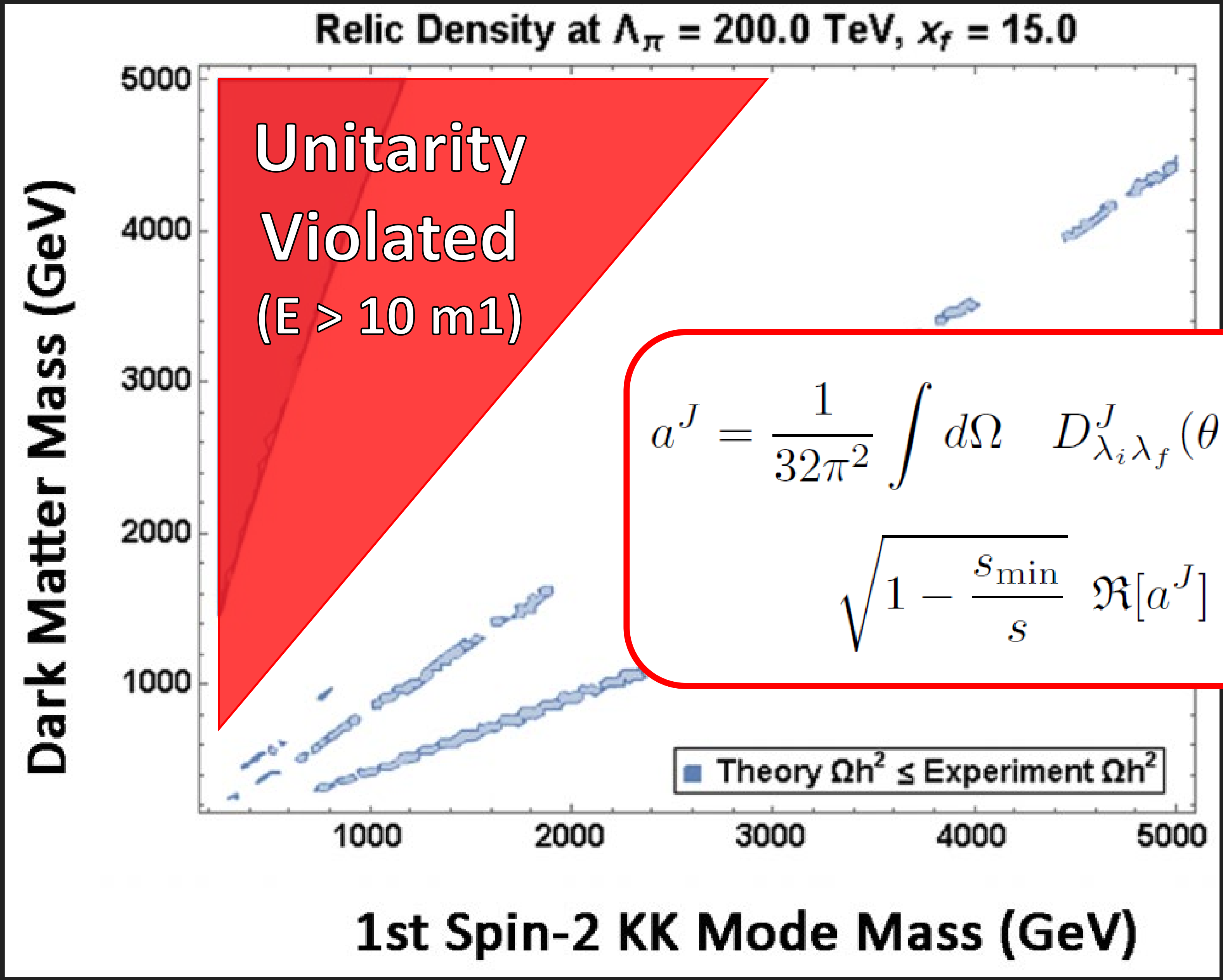

$$a^J = \frac{1}{32\pi^2}\int d\Omega \quad D^J_{\lambda_i \lambda_f}(\theta,\phi)\mathcal{M}(s,\theta,\phi)$$

$$\sqrt{1-\frac{s_{\rm min}}{s}}\;\Re[a^J] \le \frac{1}{2}$$

Key

Allowed

Dark Matter Overabundant

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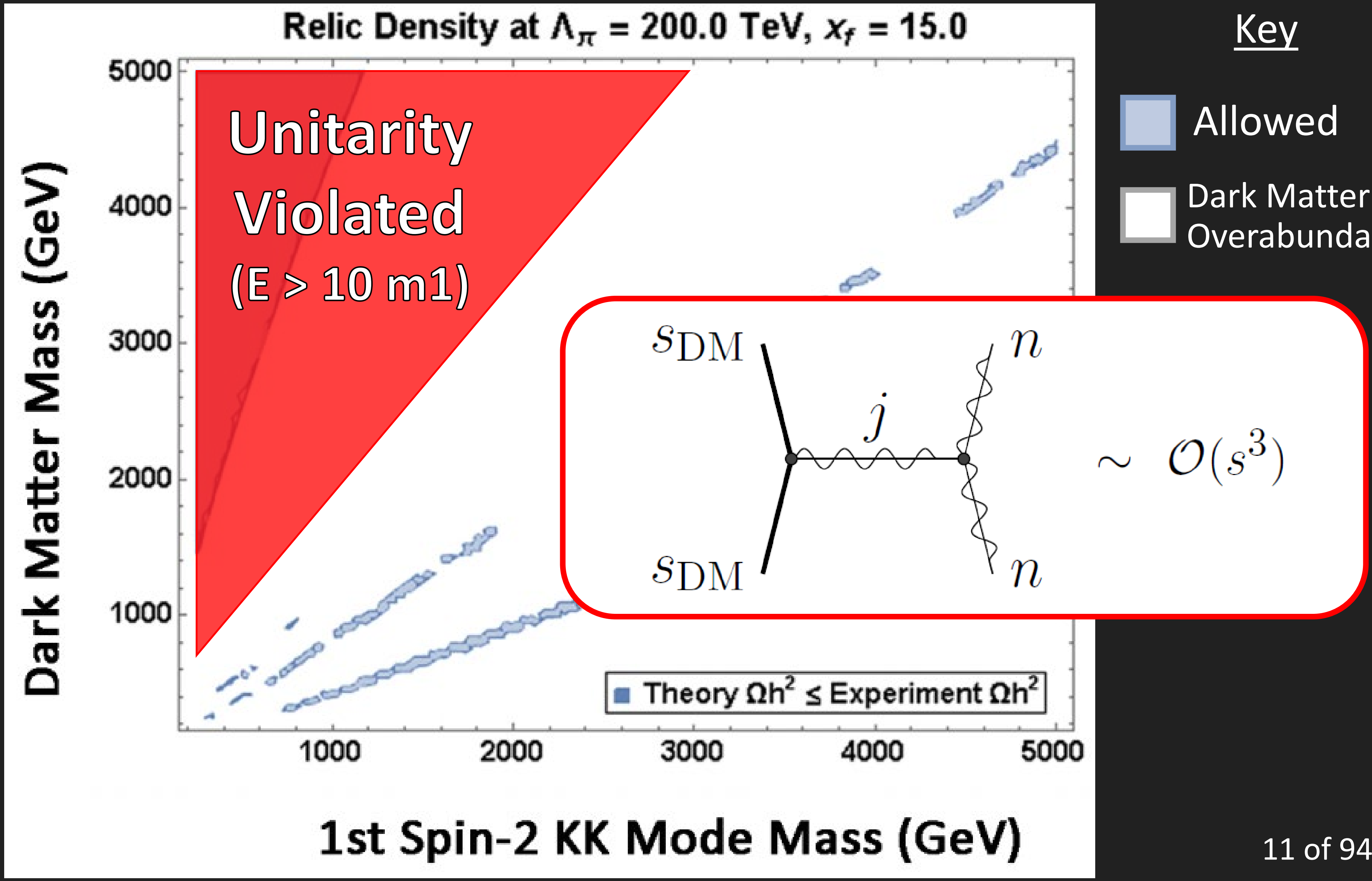


Key

Allowed

Dark Matter Overabundant

Present Day

On the work I'm presenting today

My Collaborators at MSU & UCSD:

R. Sekhar Chivukula

Kirtimaan A. Mohan

Dipan Sengupta

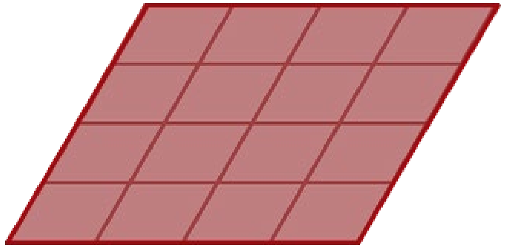
Elizabeth H. Simmons

- **Our Relevant Papers:**
  - "Sum Rules for Massive Spin-2 Kaluza-Klein Elastic Scattering Amplitudes" [arXiv:1910.06159], Phys. Rev. D 100, 115033
  - "Scattering Amplitudes of Massive Spin-2 Kaluza-Klein States Grow Only as O(s)" [arXiv:1906.11098] , Phys. Rev. D 101, 055013
  - "Massive Spin-2 Scattering Amplitudes in Extra-Dimensional Theories" [arXiv: 2002.12458], Phys. Rev. D 101, 075013

Extra-Dimensional Gravity Models
(ONLY gravity; no matter)

4D Spacetime → 4D Graviton

Flat 4D Spacetime



$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$ds^2 = dt^2 - d\vec{x}^{\,2} = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

4D Spacetime → 4D Graviton

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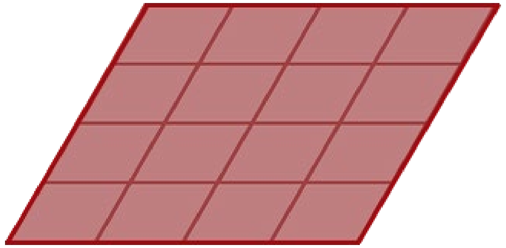
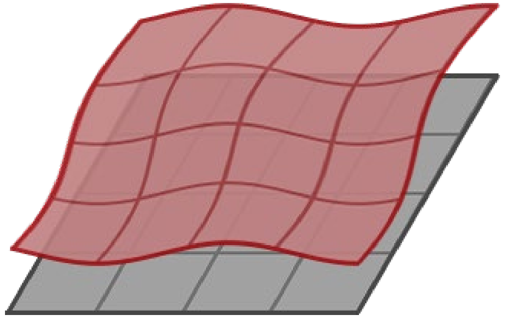


Nearly Flat Spacetime



$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\frac{2}{M_{\mathrm{Pl}}}\,\hat{h}^{(0)}_{\mu\nu}}_{\text{disturbance}}$$

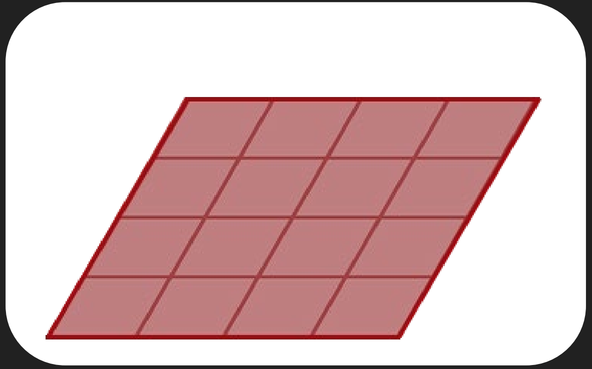
M_{Pl} = the reduced Planck mass

$$ds^2 = dt^2 - d\vec{x}^{\,2} = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

4D Spacetime → 4D Graviton

Flat 4D Spacetime



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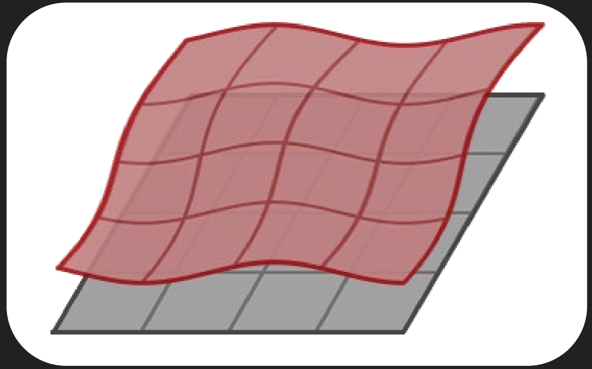
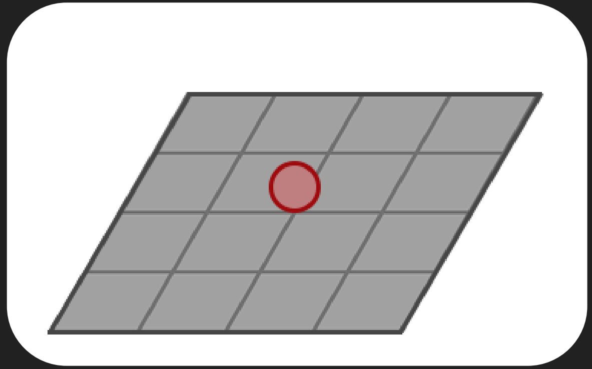


$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\frac{2}{M_{\rm Pl}}\, \hat{h}^{(0)}_{\mu\nu}}_{\text{disturbance}}$$

M_{Pl} = the **reduced Planck mass**

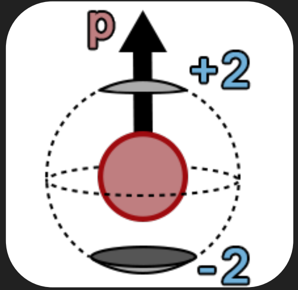
$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

Graviton



$$\hat{h}^{(0)}_{\mu\nu} = \text{4D massless spin-2 field}$$
$$\equiv \text{the ``4D graviton'' field}$$

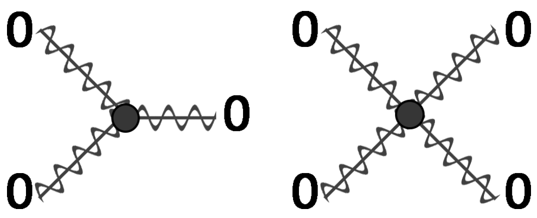
2 Possible Helicities:



2-to-2 Scattering in 4D Gravity (4DG)

$$\mathcal{L}_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2}\sqrt{g}\,R(g)$$

WFE



Einstein-Hilbert Lagrangian

Vertices

WFE = Weak Field Expansion

A perturbative expansion of the gravitational Lagrangian by assuming the curved metric is very close to a “background metric” (= a classical solution of the Einstein Field equations).

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{2}{M_{\mathrm{Pl}}}\hat{h}^{(0)}_{\mu\nu}(x)$$

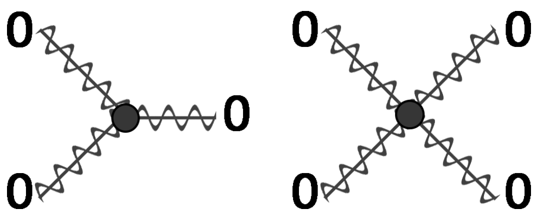
2-to-2 Scattering in 4DG

$$\mathcal{L}_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2}\sqrt{g}\,R(g)$$

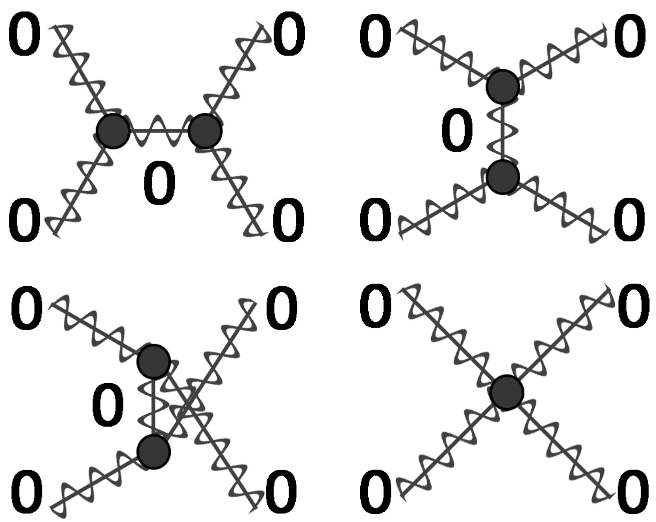
Einstein-Hilbert Lagrangian

WFE

$$\boxed{g = \eta + \frac{2}{M_{\text{Pl}}}\hat{h}^{(0)}}$$



Vertices



Tree-Level 2-to-2 Diagrams

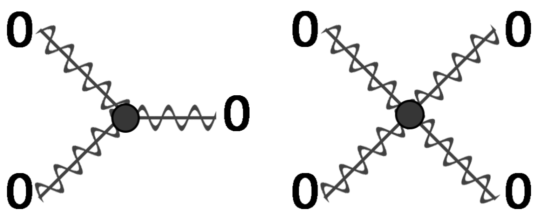
2-to-2 Scattering in 4DG

$$\mathcal{L}_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\sqrt{g}\,R(g)$$

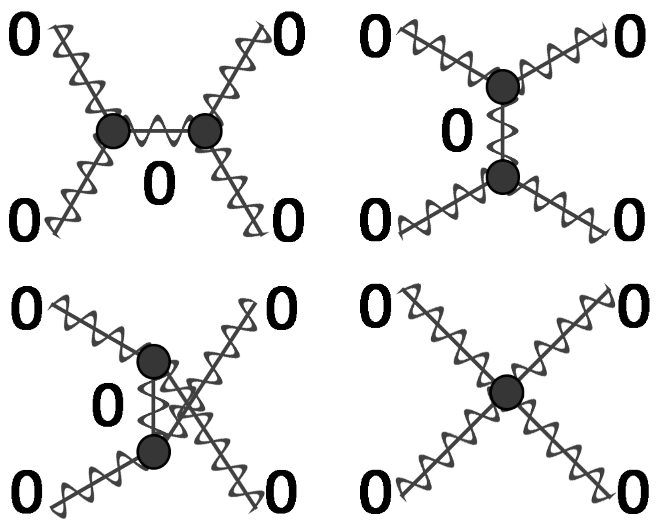
WFE

$$\boxed{g = \eta + \frac{2}{M_{\rm Pl}}\hat{h}^{(0)}}$$

Einstein-Hilbert Lagrangian

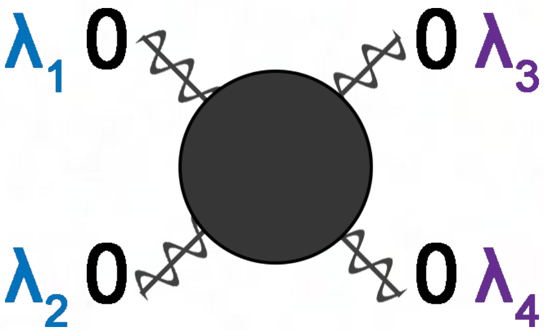


Vertices



Tree-Level 2-to-2 Diagrams

$\mathcal{M} =$



$\propto \boxed{\left(\frac{E}{M_{\rm Pl}}\right)^2}$

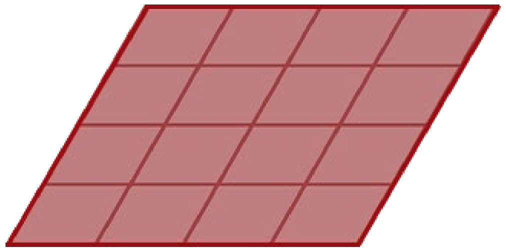
2^4 = 16 helicity combos

Could anticipate this, because...

- **M_{Pl}** is only dimensionful parameter.
- Two instances of coupling present.
- This matrix element is dimensionless.

5D Spacetime → 5D Graviton

Flat 5D Spacetime



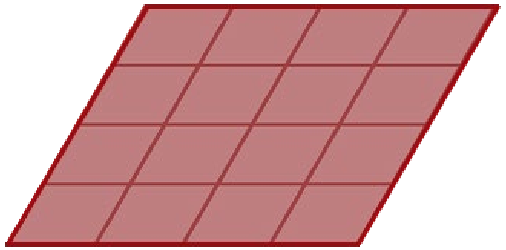
$$\eta_{MN} = \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}$$

Add extra dimension, length $\boldsymbol{\pi r_c}$

$\mathbf{r_c}$ = compactification radius

5D Spacetime → 5D Graviton

Flat 5D Spacetime



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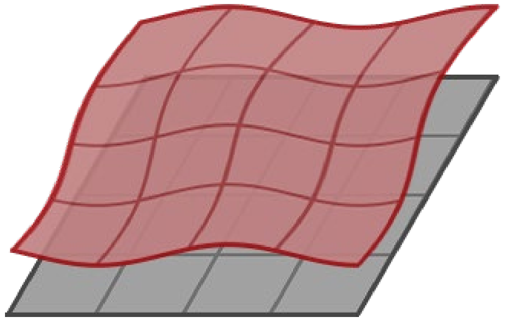
Add extra dimension, length **πr_c**

r_c = **compactification radius**

Nearly Flat Spacetime



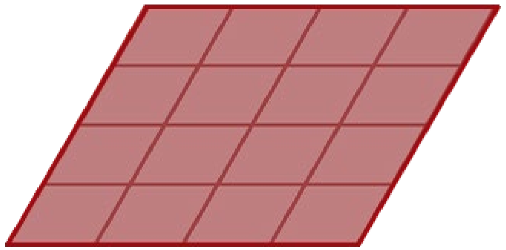
$$G_{MN} = \eta_{MN} + \underbrace{\kappa\,\hat{h}_{MN}}_{\text{disturbance}}$$

κ = **5D coupling strength** and has units **$[E]^{-3/2}$**

$$ds^2 = G_{MN}\,dx^M\,dx^N$$

5D Spacetime → 5D Graviton

Flat 5D Spacetime



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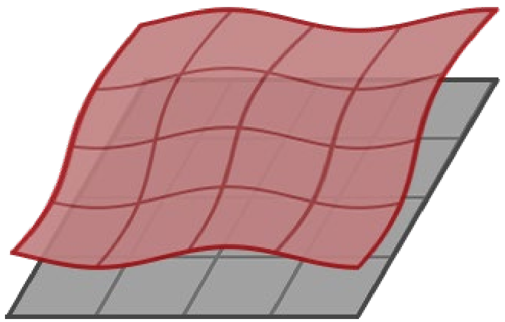
Nearly Flat Spacetime



$$G_{MN} = \eta_{MN} + \underbrace{\kappa\,\hat{h}_{MN}}_{\text{disturbance}}$$

κ = 5D coupling strength
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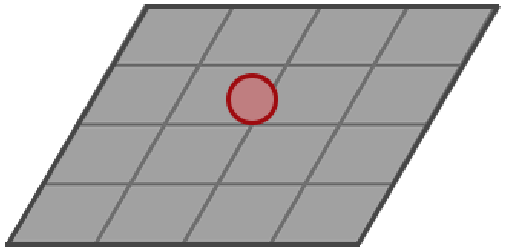
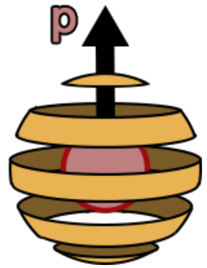
$$ds^2 = G_{MN}\,dx^M\,dx^N$$

Graviton



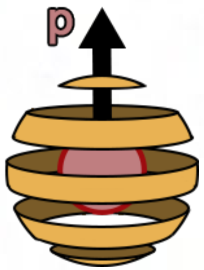
$$\hat{h}_{MN} \equiv \text{the “5D graviton” field}$$

5 Possible Helicities:



5D Spacetime → 5D Graviton

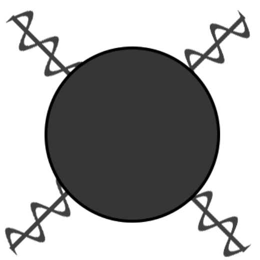
$\hat{h}_{MN}$



Massless 5D Graviton

$\mathcal{L}_{5D}$

High-Energy 5D Behavior



$\sim (\kappa^2 E^2)$

Kaluza-Klein Decomposition: What is it?

… but what if we did the calculation in 4D instead?

Consider:

Kaluza-Klein Decomposition

- Introduced to unify Gravity + E&M (cool idea, didn't work)
- Sometimes called "mode expansion."
- Like a fancier version of Fourier decomposition.

**** IMPORTANT ****

The following is moreso a mnemonic.
Many details have been omitted.
(e.g. the warping in RS1)

Kaluza-Klein Decomposition: What is it?

$$\boxed{G_{MN} = \eta_{MN} + h_{MN}}$$

5D Lorentz transformations
contain
4D Lorentz transformations

$$\Rightarrow \quad h_{MN} \approx \begin{bmatrix} \begin{array}{c} h_{\mu\nu} \\ \text{spin-2} \end{array} & \begin{array}{c} \rho_{\mu} \\ \text{spin-1} \end{array} \\ \begin{array}{c} \rho_{\nu} \\ \text{spin-1} \end{array} & \begin{array}{c} r \\ \text{spin-0} \end{array} \end{bmatrix}$$

(not independent)

Caution:

$$h_{\mu\nu} = h_{\mu\nu}(x, y)$$

KK Decomposition: vs Fourier Decomposition

1 continuous variable

$$f(y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} f^{(n)} \; \psi_n(\varphi)$$

1 discrete index

Boundary Condition

$$\left(\partial_y \psi_n\right)\Big|_{y=0} = \left(\partial_y \psi_n\right)\Big|_{y=L} = 0$$

Differential Equation

$$\boxed{\left(\partial_y^2 \psi_n\right) = -\left(\frac{n\pi}{L}\right)^2 \psi_n}$$

Solutions

$$\boxed{\psi_n = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ \cos\left(\frac{n\pi y}{L}\right) & n > 0 \end{cases}}$$

Normalization

$$\frac{2}{L} \int_0^L dy \;\; \psi_m \psi_n = \delta_{m,n}$$

KK Decomposition: 5D → 4D

$$f(x,y) \sim f^{(n)}(x)\,\psi_n(\varphi)$$

$$\begin{cases} \Box_{5\mathrm{D}} f = 0 \\ \Box_{4\mathrm{D}} f^{(n)} = -m_n^2 f^{(n)} \end{cases} \Longrightarrow \partial_\varphi^2 \psi_n = -(m_n r_c)^2 \psi_n$$

Differential Equation

(again: RS1 more complicated)

1 continuous variable

$$f(x,y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} f^{(n)}(x)\,\psi_n(\varphi)$$

1 discrete index

KK Decomposition: 5D → 4D

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Differential Equation

(again: RS1 more complicated)

1 continuous variable

$$f(x,y) = \frac{1}{\sqrt{L}} \quad (\varphi)$$

4D Field

$$f^{(n)}(x)$$

KK Index = n

mass = m_n

1 discrete index

KK Decomposition: 5D → 4D

$$f(x,y) \sim f^{(n)}(x)\,\psi_n(\varphi)$$

$$\begin{cases} \Box_{5\text{D}} f = 0 \\ \Box_{4\text{D}} f^{(n)} = -m_n^2 f^{(n)} \end{cases} \Longrightarrow \quad \partial_\varphi^2 \psi_n = -(m_n r_c)^2 \psi_n$$

Differential Equation

(again: RS1 more complicated)

1 continuous variable

$$f(x,y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} f^{(}$$

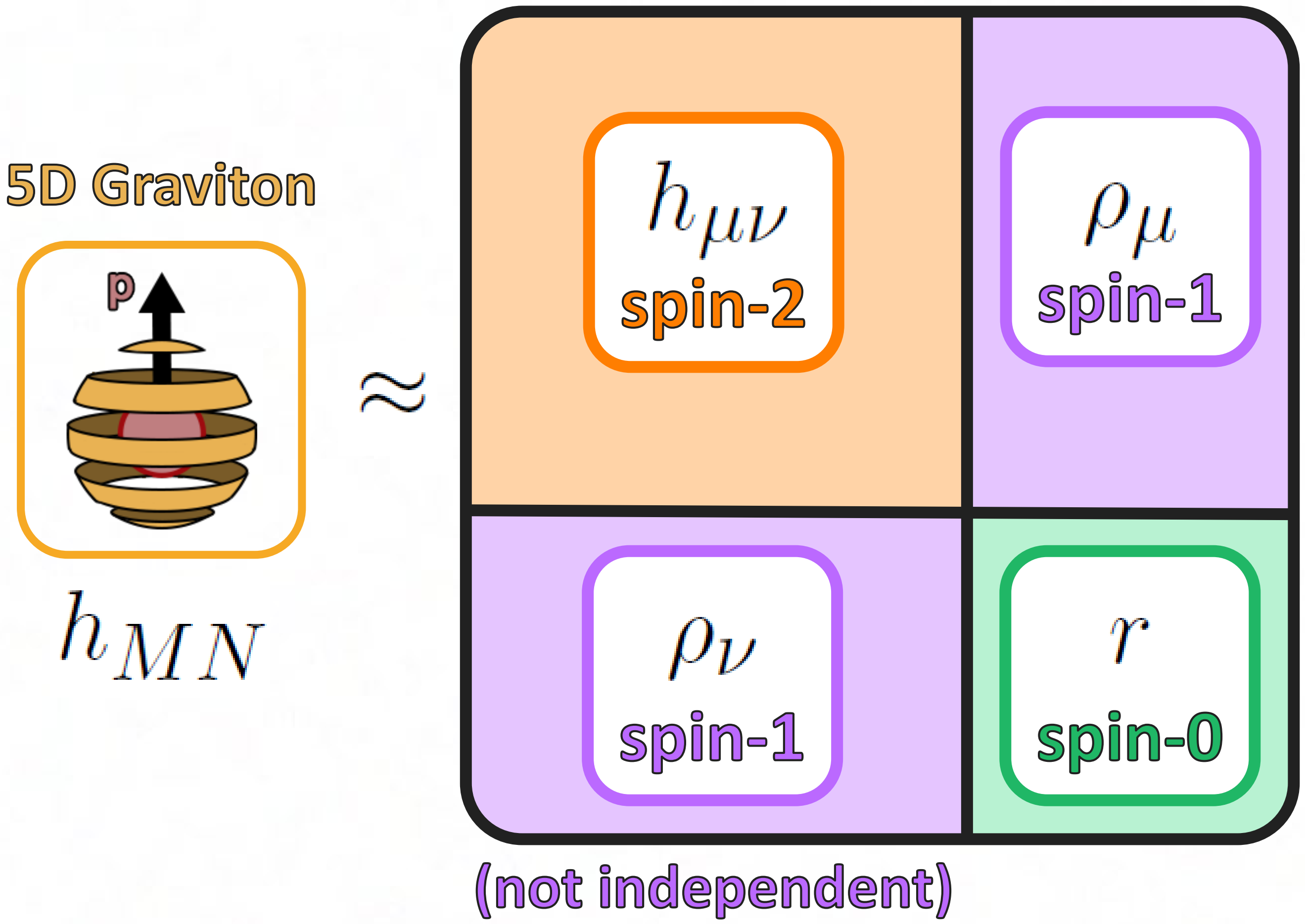
Wavefxn

$$\psi_n(\varphi)$$

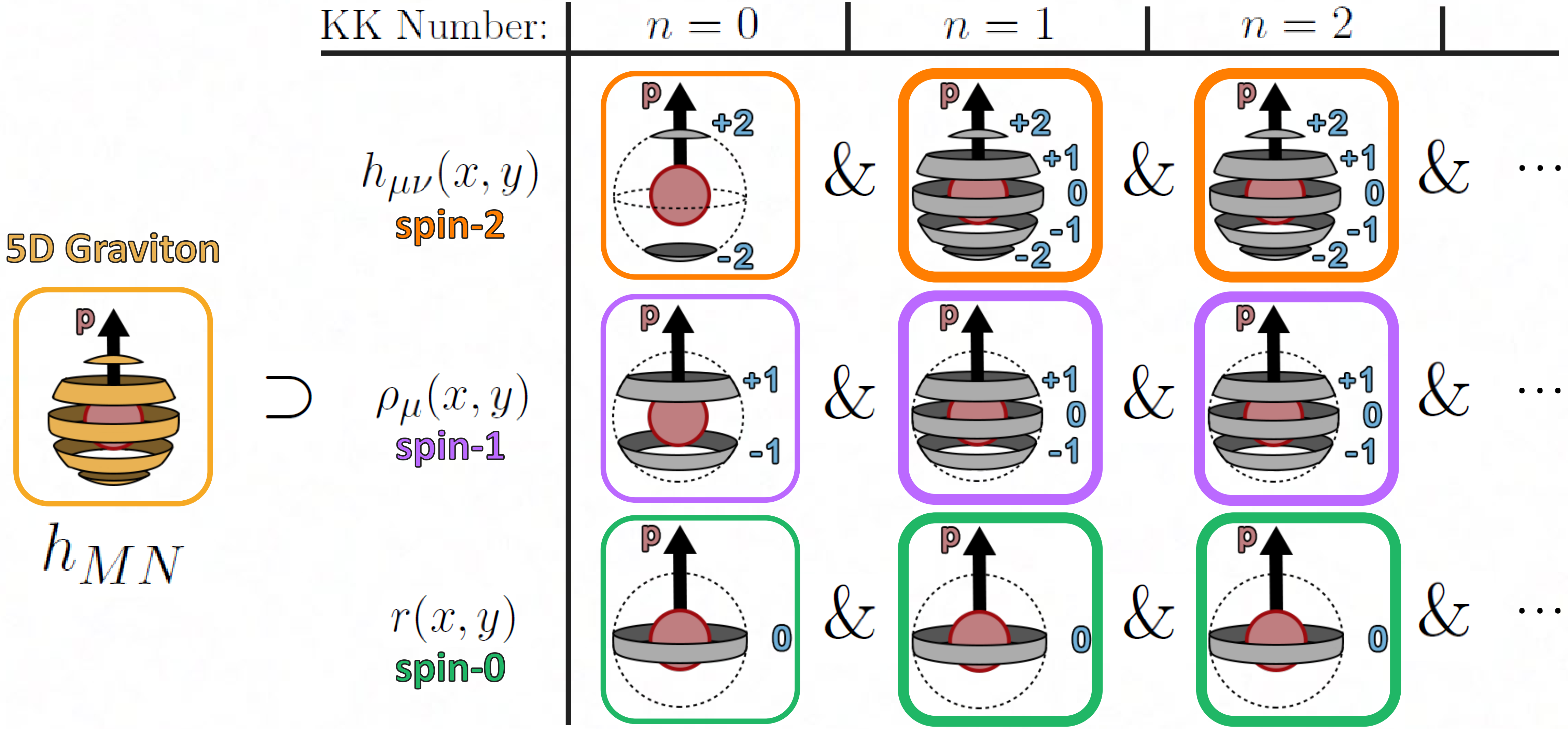
KK Index = n

l discrete index

KK Decomposition: Apply to 5D Graviton



KK Decomposition: 5D Graviton → 4D Particles



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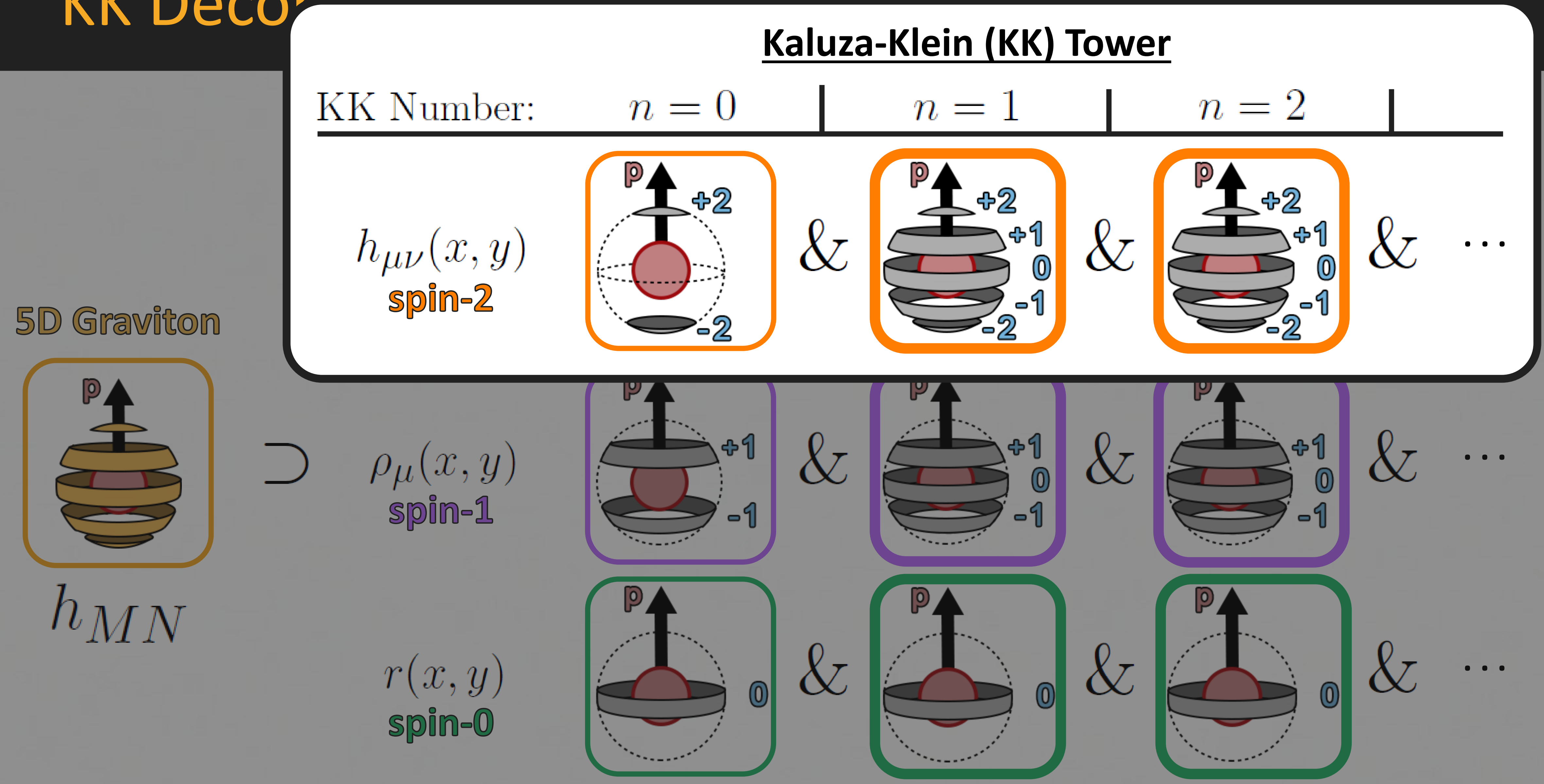

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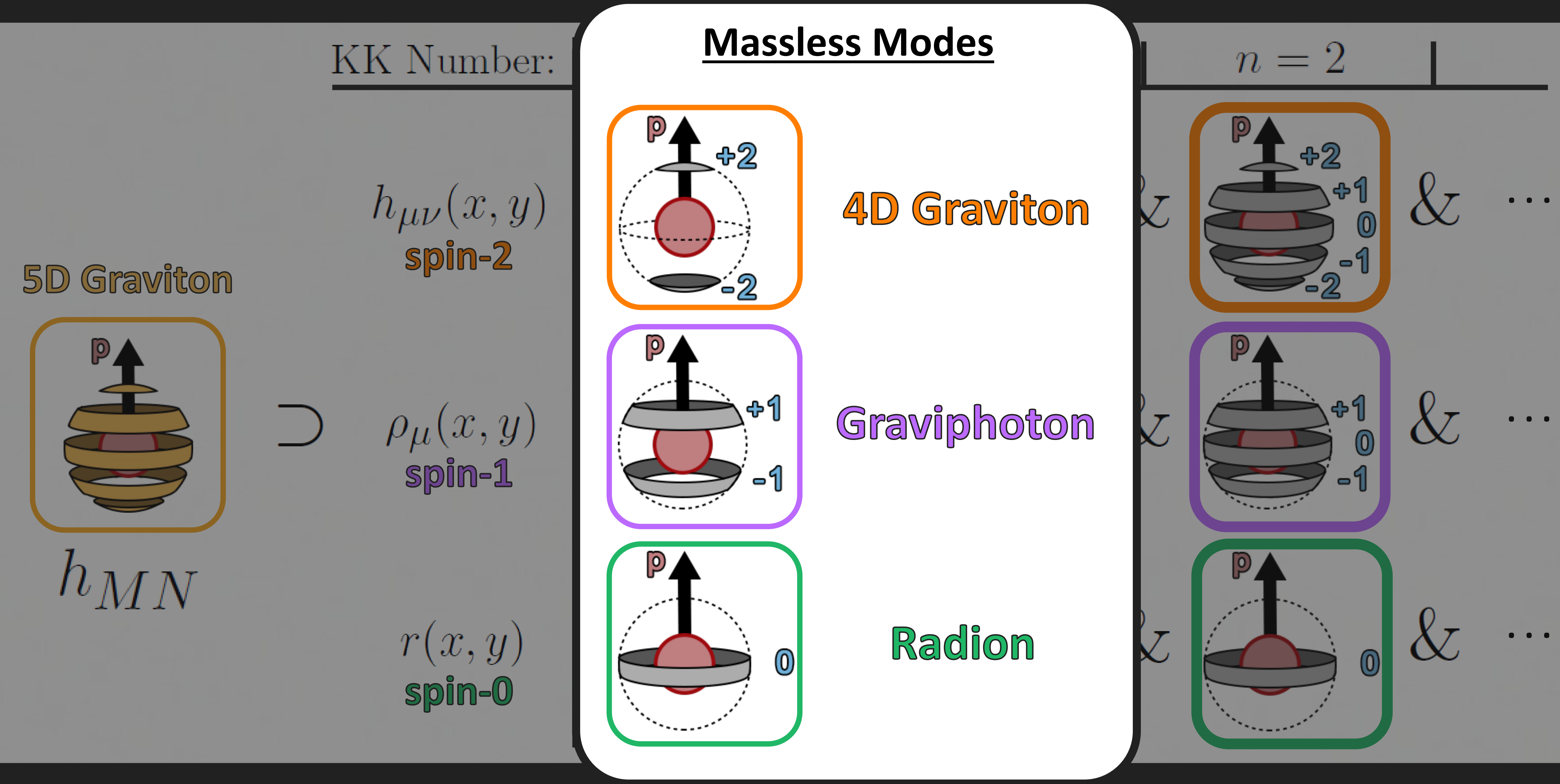

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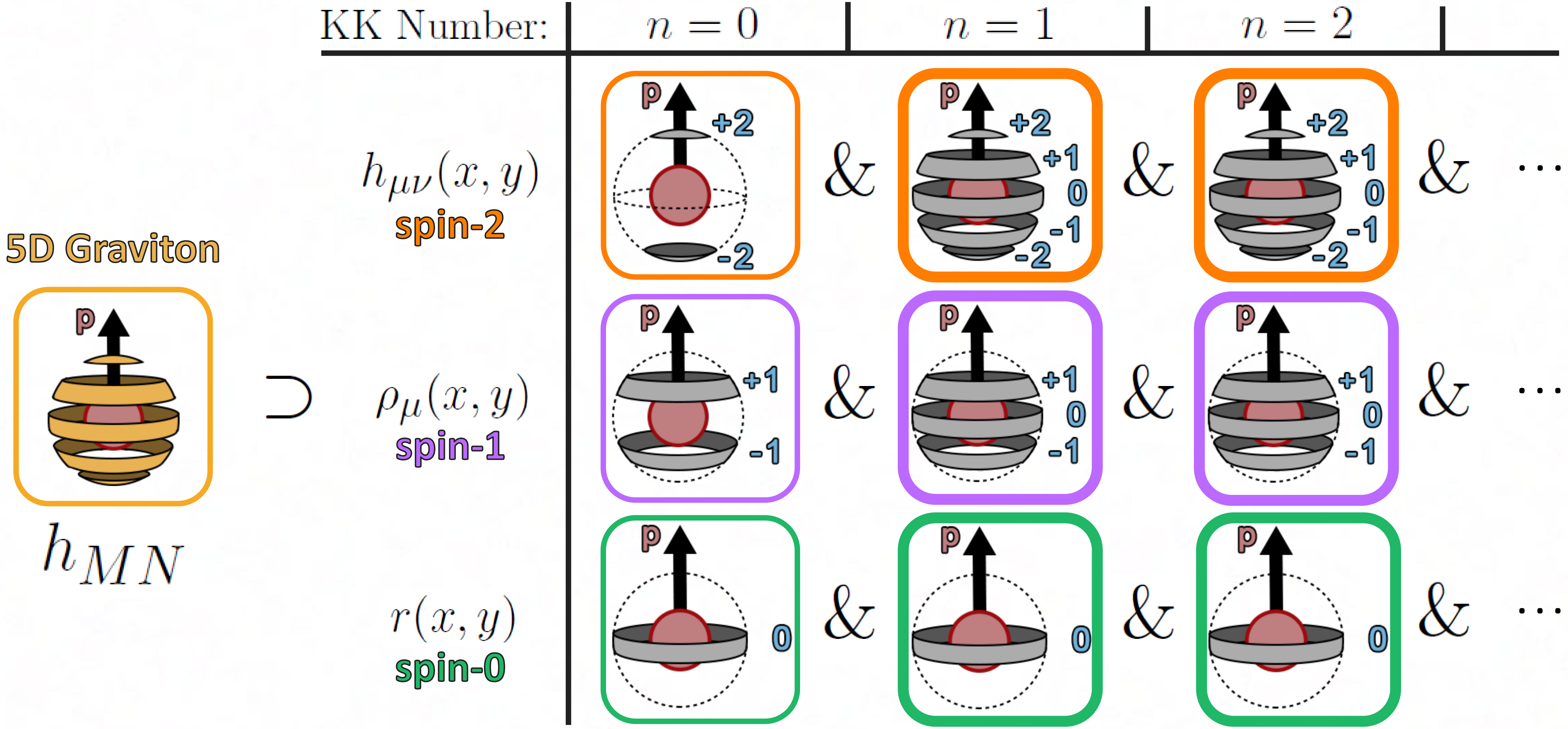

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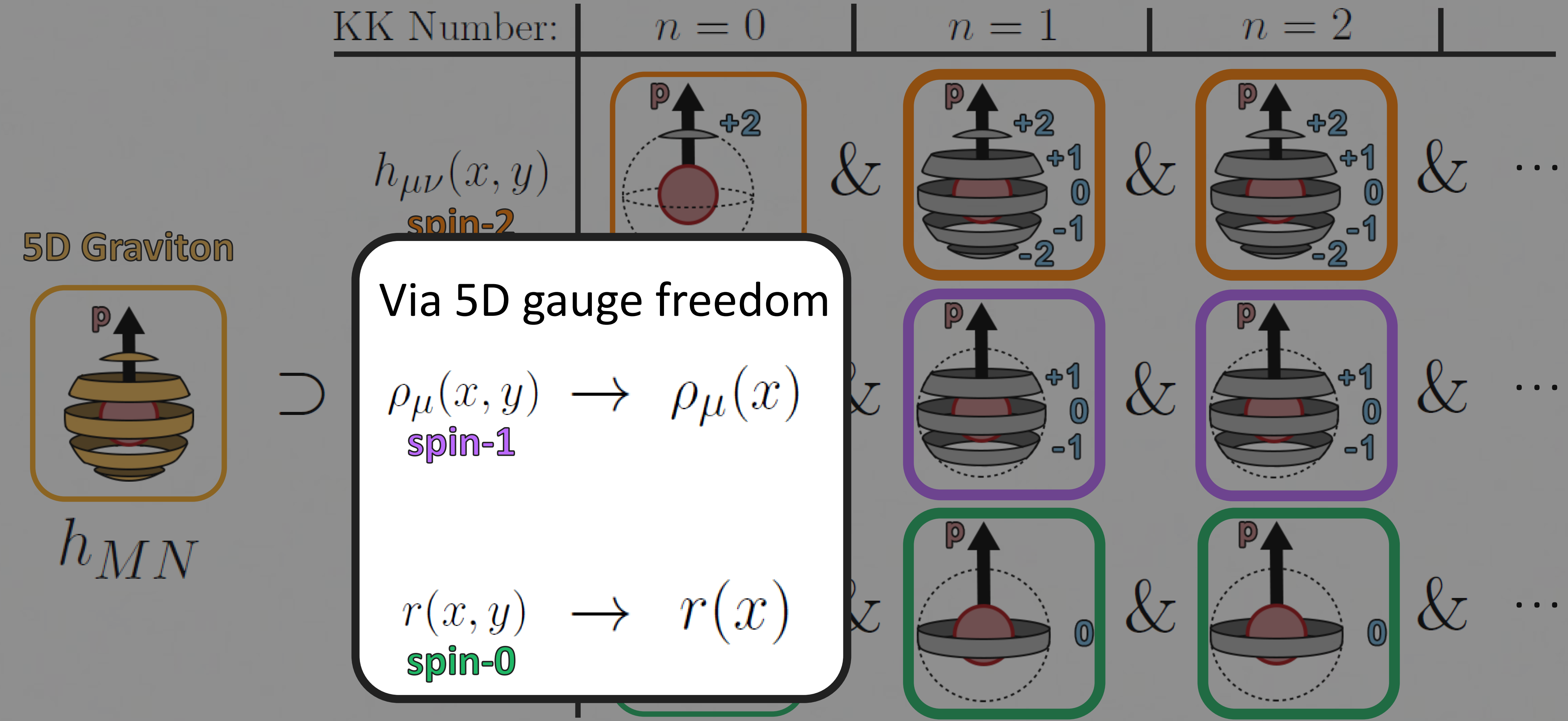


Via 5D gauge freedom

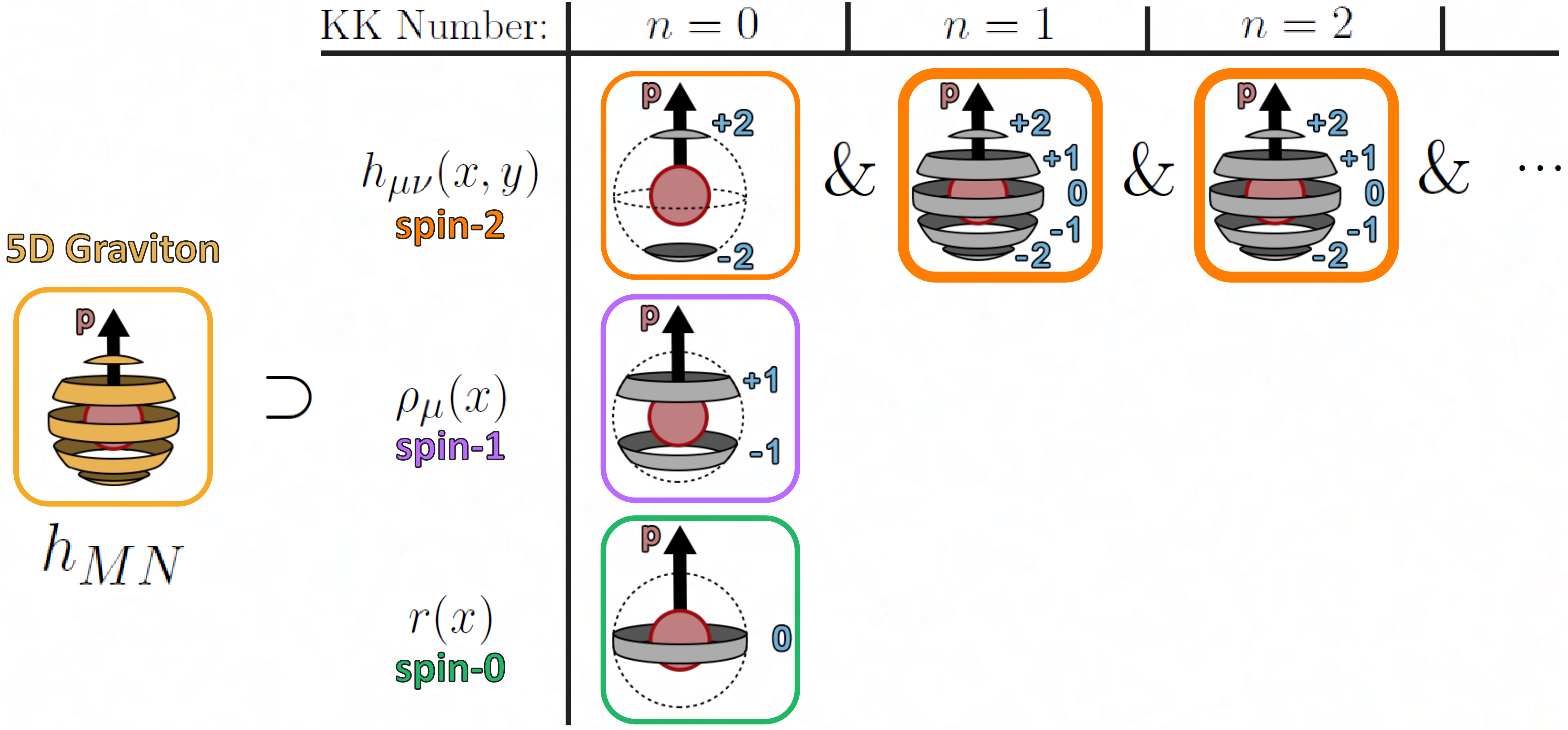
$$\rho_\mu(x,y) \rightarrow \rho_\mu(x)$$

spin-1

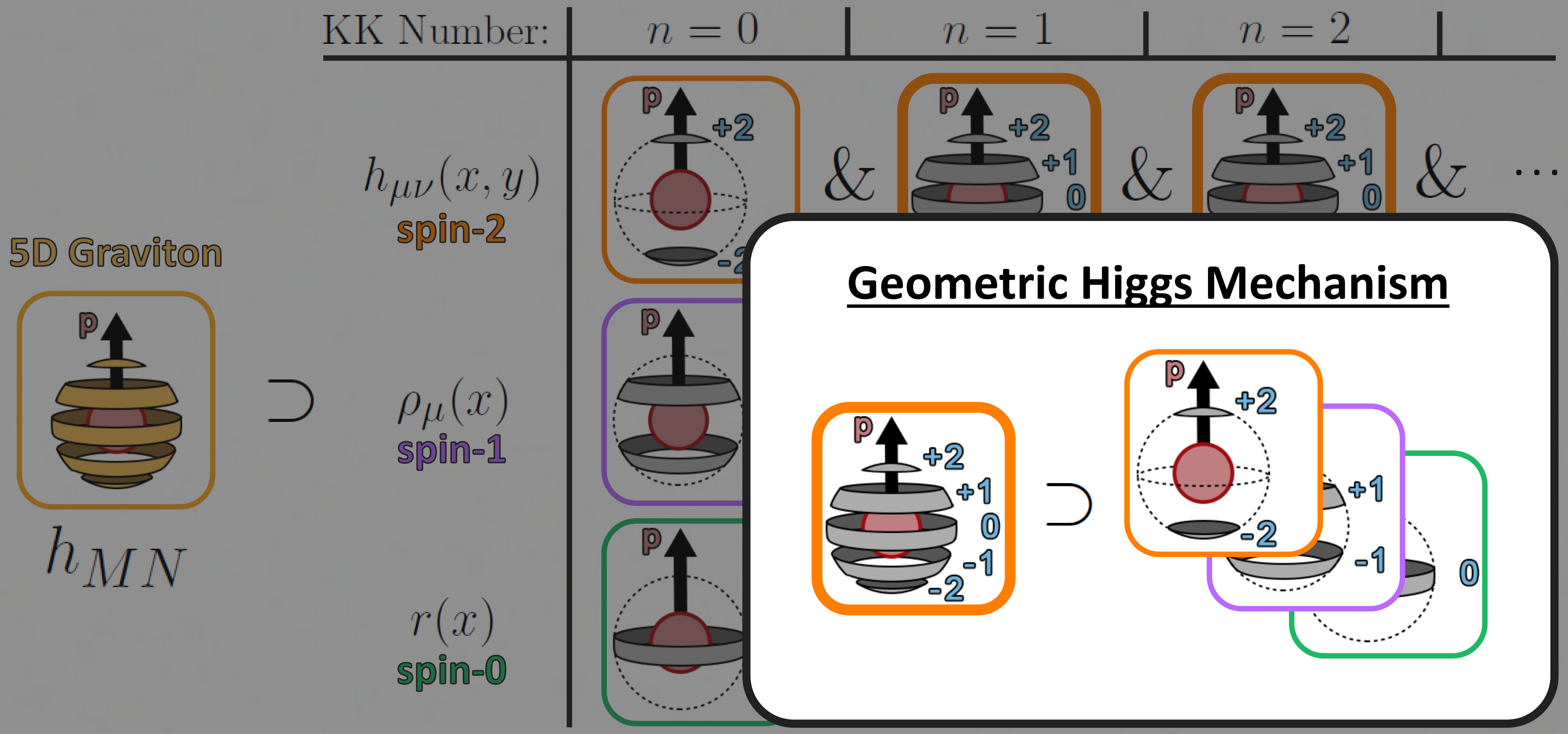
$$r(x,y) \rightarrow r(x)$$

spin-0

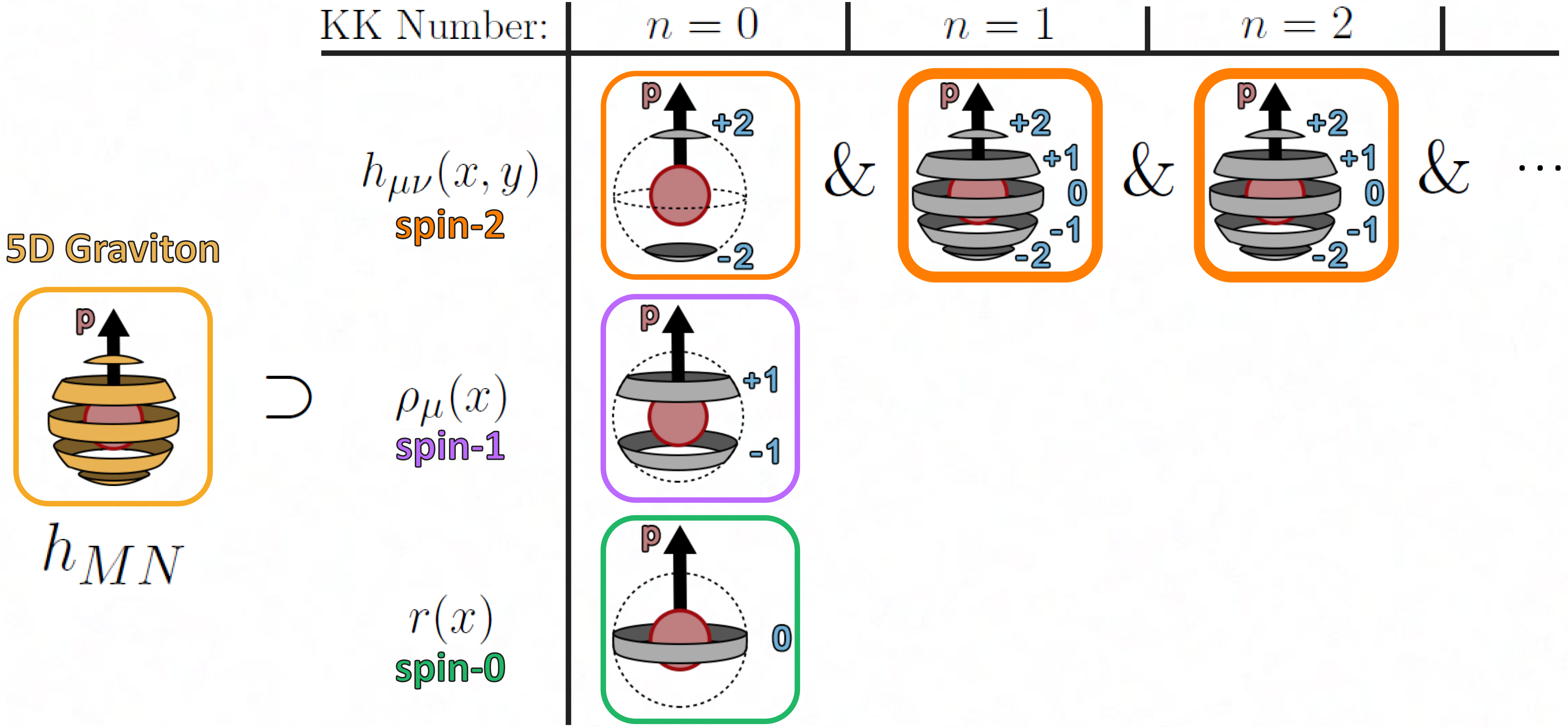
KK Decomposition: 5D Graviton → 4D Particles



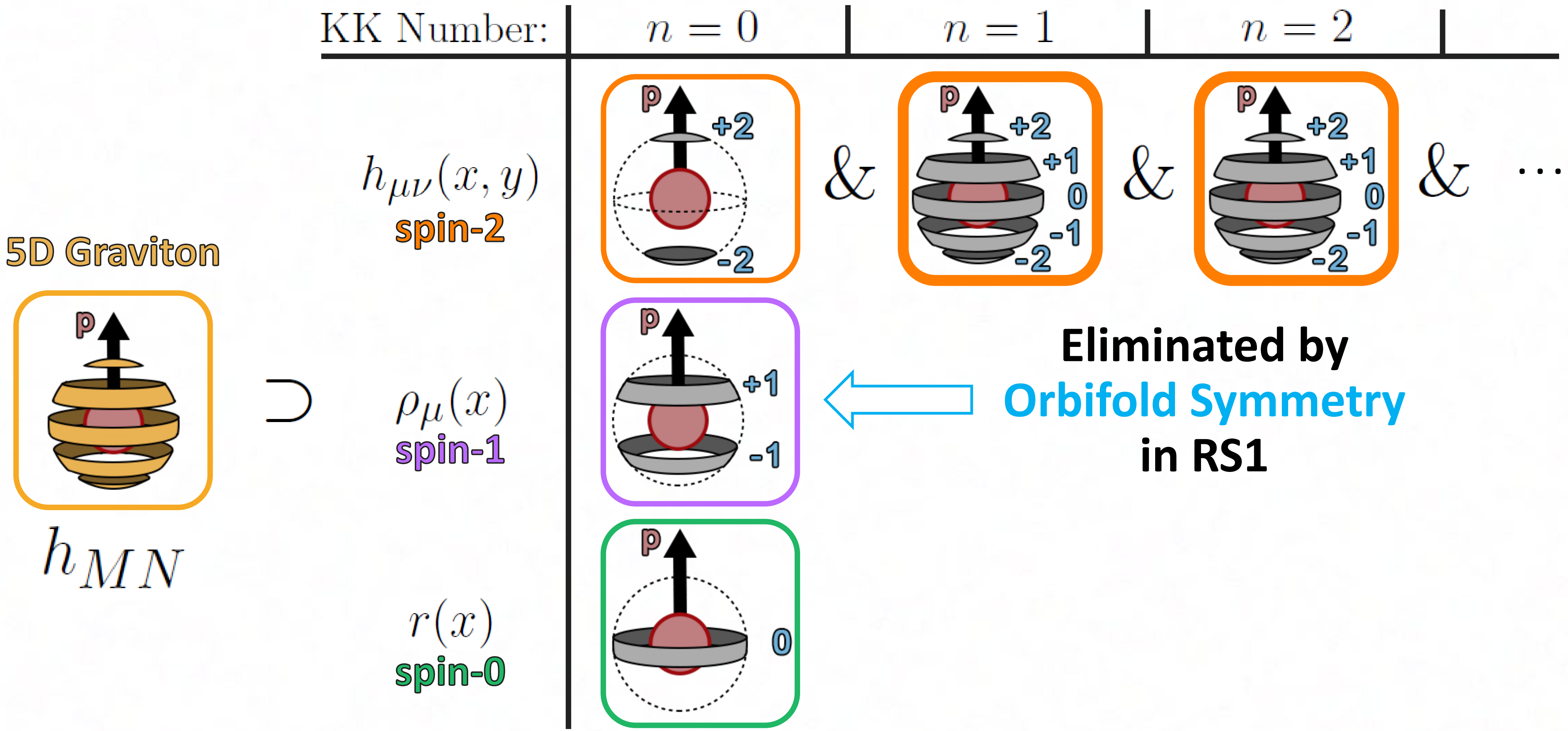
KK Decomposition: 5D Graviton → 4D Particles



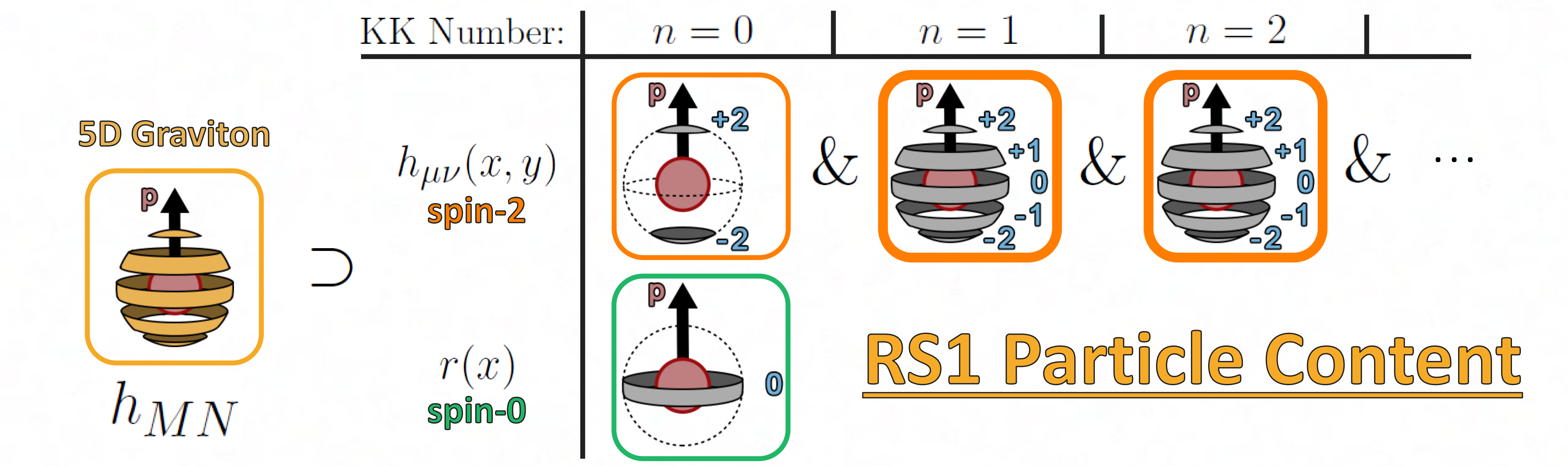
KK Decomposition: 5D Graviton → 4D Particles



KK Decomposition: 5D Graviton → 4D Particles



KK Decomposition: 5D Graviton → 4D Particles



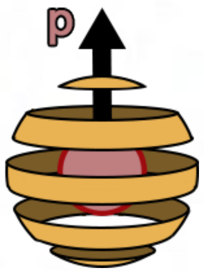
$$\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}^{(n)}_{\mu\nu}(x)\, \underline{\psi_n(\varphi)}$$

$$\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}}\, \hat{r}^{(0)}(x)\, \underline{\psi_0}$$

(more about RS1 wavefunctions soon)

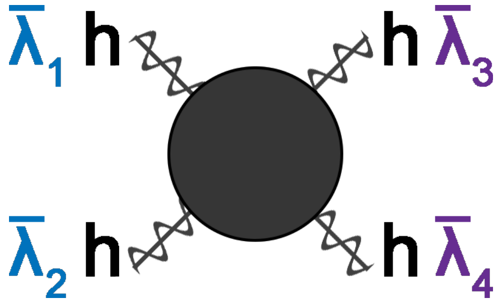
Kaluza-Klein Decomposition: 5D restricts 4D

$\hat{h}_{MN}$



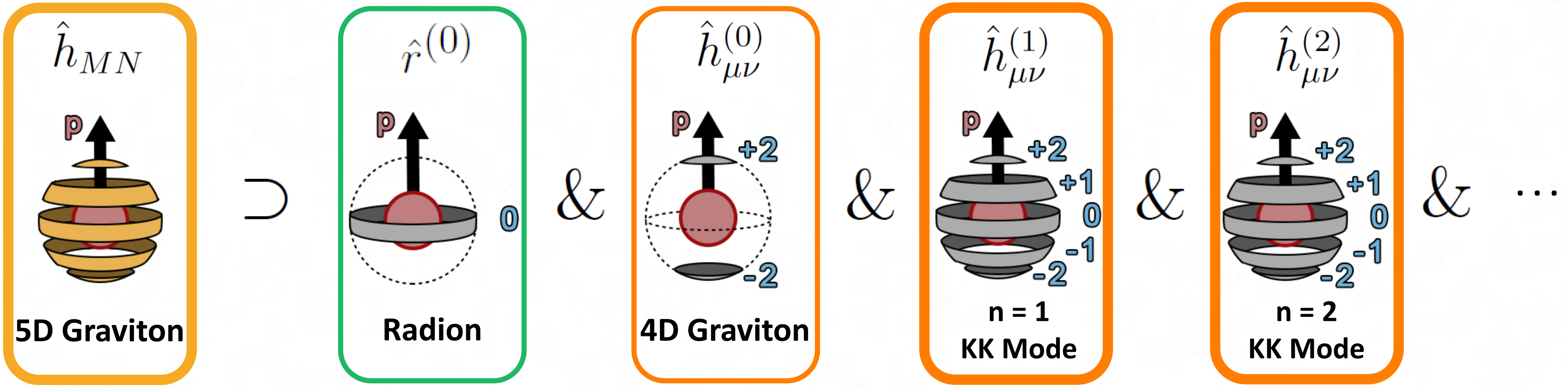
5D Graviton

High-Energy 5D Behavior

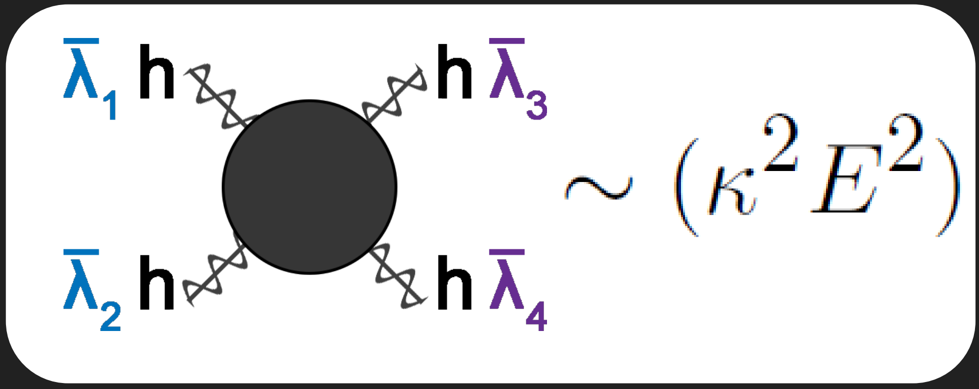


$$\sim \left(\kappa^2 E^2\right)$$

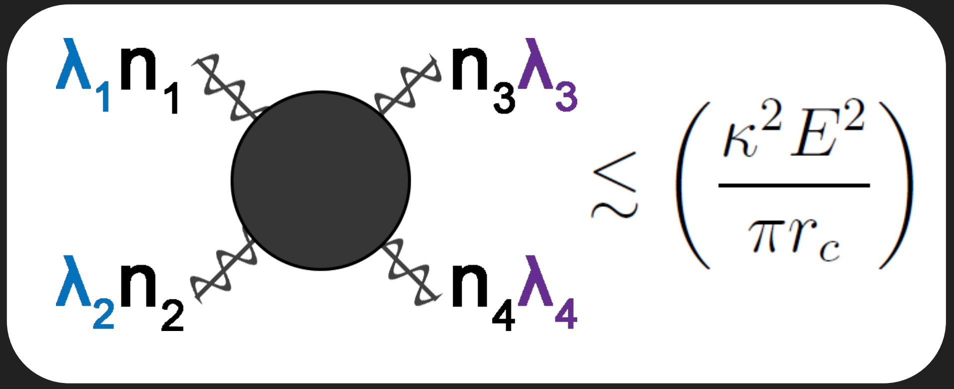
Kaluza-Klein Decomposition: A Lead?



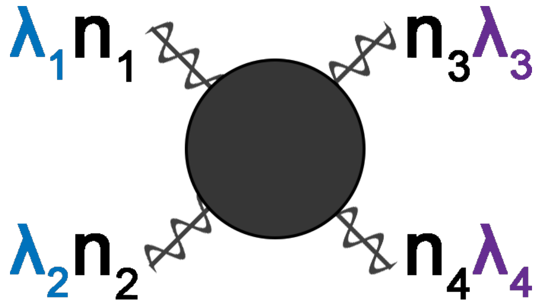
High-Energy 5D Behavior



High-Energy 4D Behavior



High-Energy 4D Behavior



$$\lesssim \left(\frac{\kappa^2 E^2}{\pi r_c}\right)$$

- We examine how this constraint impacts the **5D Randall-Sundrum (RS1) model**, which possesses warping in the extra dimension.

- We perform the **full calculation**, i.e. without approximations. No "large warping" limit here!

- By **demanding all energy growth faster than O(s) to vanish**, we derive **sum rules** relating couplings & masses between KK modes (… *and then prove most!*)

The Randall-Sundrum "RS1" Model

The Original RS Papers: [arXiv:hep-ph/9905221 and 9906064]

Building RS1

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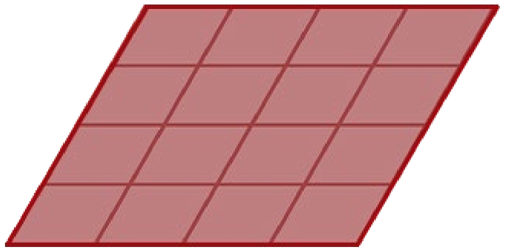
N/A

Flat 4D Spacetime



$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

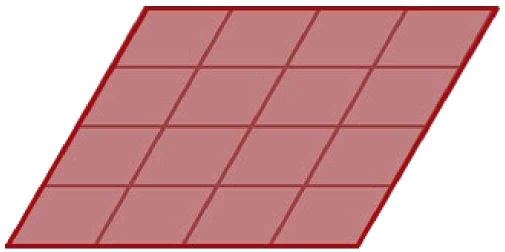
$$ds^2 = dt^2 - d\vec{x}^{\,2} = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

Begin with **flat 4D spacetime**.

Building RS1

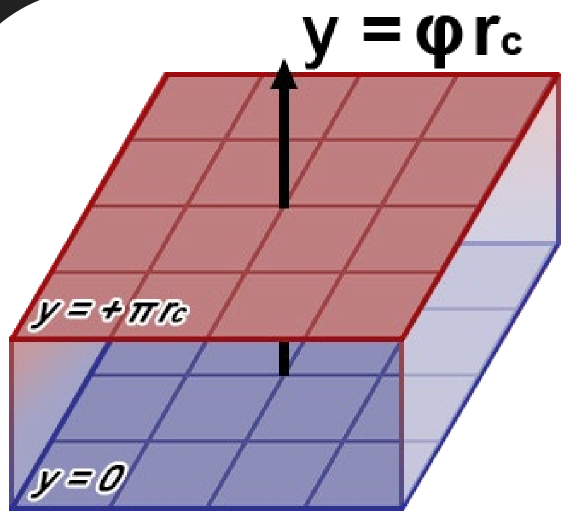
Previous Slide

Flat 4D Spacetime



$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Add a Finite Extra Dimension



$$\eta_{MN} = \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}$$

$$ds^2 = dt^2 - d\vec{x}^2 - dy^2$$

$y \in [0,\pi r_c]$ where **r_c =** **compactification radius**

Often we replace **y** with **φ = $y/r_c \in [0,\pi]$**

Building RS1

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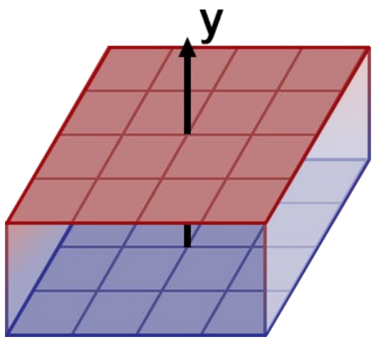
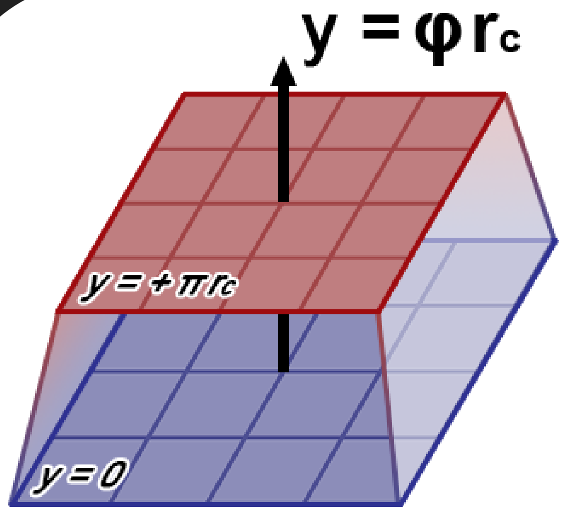
Add Extra Dimension



$$\eta_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

where $x^M = (x^\mu, y)$

Warp Spacetime



$$\eta^{(RS)}_{MN} = \begin{pmatrix} \boxed{\varepsilon^{-2}\,\eta_{\mu\nu}} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\boxed{\varepsilon \equiv e^{+kr_c|\varphi|}}$

k = warping parameter

k = 0: flat extra dimension

k > 0: needs brane tensions, cosmological constant

Building RS1

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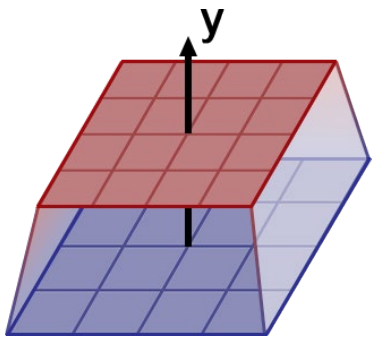
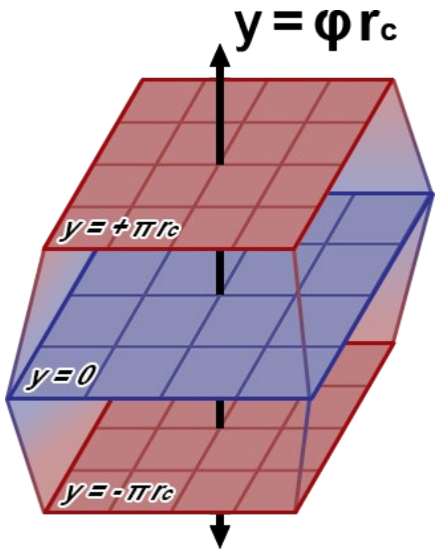
Warp Spacetime



$$\eta^{(\mathrm{RS})}_{MN} = \begin{pmatrix} \varepsilon^{-2}\,\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{where} \quad \varepsilon \equiv e^{+kr_c|\varphi|}$$

Orbifold



Spacetime Symmetries:

$$\begin{cases} \varphi \to -\varphi \\ \varphi \to \varphi + 2\pi \end{cases}$$

Separates orbifold even states (including the graviton + radion) from orbifold odd states (the graviphoton, which is unimportant to us)

Building RS1

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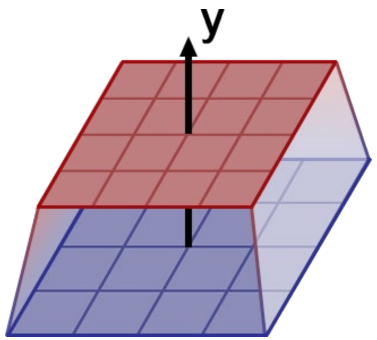
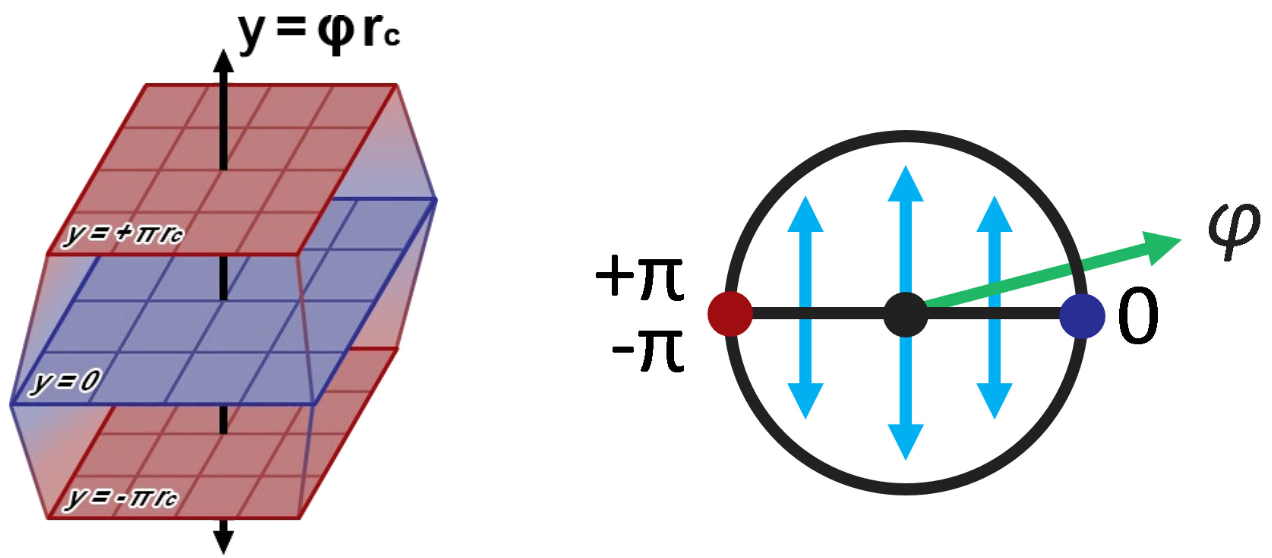
Warp Spacetime



$$\eta_{MN}^{(\mathrm{RS})} = \begin{pmatrix} \varepsilon^{-2}\,\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{where} \quad \varepsilon \equiv e^{+kr_c|\varphi|}$$

Orbifold



Separates **orbifold even states** (including the **graviton** + **radion**) from **orbifold odd states** (the **graviphoton**, which is unimportant to us)

Building RS1

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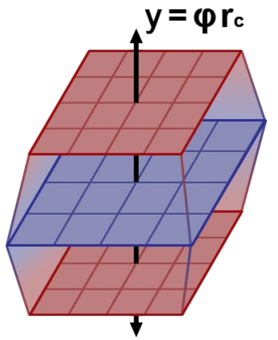
Orbifold



$$\eta^{(\mathrm{RS})}_{MN} = \begin{pmatrix} \varepsilon^{-2}\,\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\quad \varepsilon \equiv e^{+kr_c|\varphi|}$

The RS1 Metric & Field Content

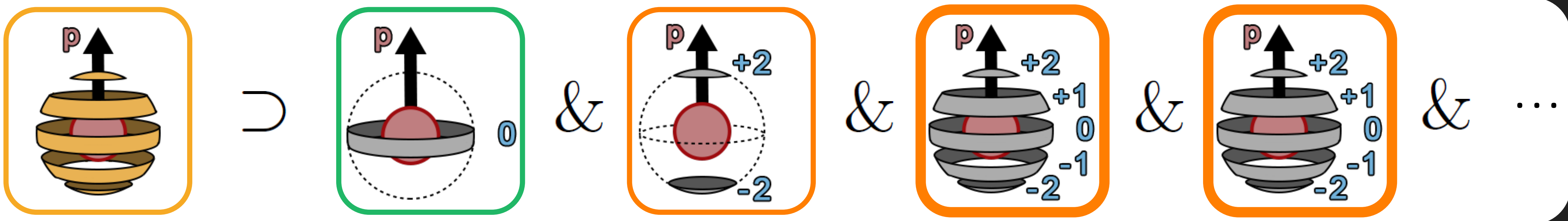
$$\eta^{(\mathrm{RS})}_{MN} \xrightarrow{\text{perturb}} G^{(\mathrm{RS})}_{MN} = \begin{pmatrix} \varepsilon^{-2}\, g_{\mu\nu}\, e^{-2\hat{u}} & 0 \\ 0 & -(1+2\hat{u})^2 \end{pmatrix}$$

where

$$\boxed{g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa \hat{h}_{\mu\nu}} \qquad \text{and} \qquad \boxed{\hat{u} \equiv \frac{\kappa}{2\sqrt{6}}\left[\varepsilon^{+2}\, e^{-\pi k r_c}\, \hat{r}\right]}$$

$$\begin{aligned} \hat{h}_{\mu\nu}(x,y) &= \text{massless 5D spin-2} \\ \hat{r}(x) &= \text{massless 5D spin-0} \quad \text{(note it's } y\text{-independent!)} \end{aligned}$$

Building RS1: Kaluza-Klein (KK) Decomposition



$$\boxed{\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}(x)\, \psi_0}$$ = (**radion**)

$$\boxed{\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}^{(n)}_{\mu\nu}(x)\, \psi_n(\varphi)}$$ = (**4D graviton**) + (**massive 4D spin-2 KK modes**)

- The **wavefunctions** ψ_n solve a specific **Sturm-Liouville Equation (S-L Eq.)**
- The solution set { ψ_n } and spectrum { $m_n r_c$ } are **entirely determined by kr_c**.

Reparameterize: $(k, r_c, \kappa) \rightarrow (kr_c, m_1, M_{Pl})$

$$\begin{aligned} kr_c &= k \cdot r_c \\ m_1 &\equiv \frac{1}{r_c}\left[(m_1 r_c)\big|_{kr_c}\right] \quad \text{via S-L Eq.} \\ M_{\text{Pl}} &\equiv \frac{2}{\kappa\sqrt{k}}\sqrt{1-e^{-2kr_c\pi}} \end{aligned}$$

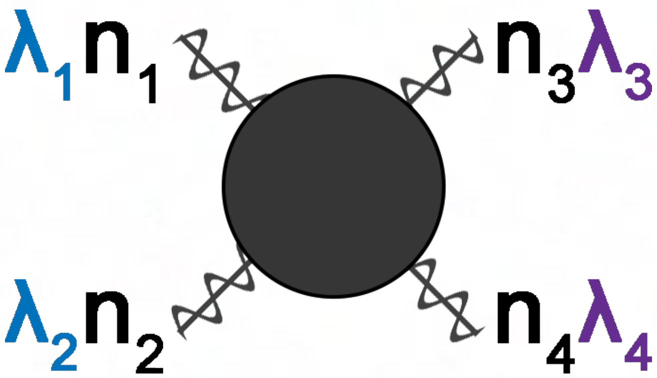
For the remainder of this presentation, whenever I show *numerical* plots, I set:

$$M_{\text{Pl}} = 2.435 \times 10^{15}\ \text{TeV}$$
$$m_1 = 1\ \text{TeV}$$

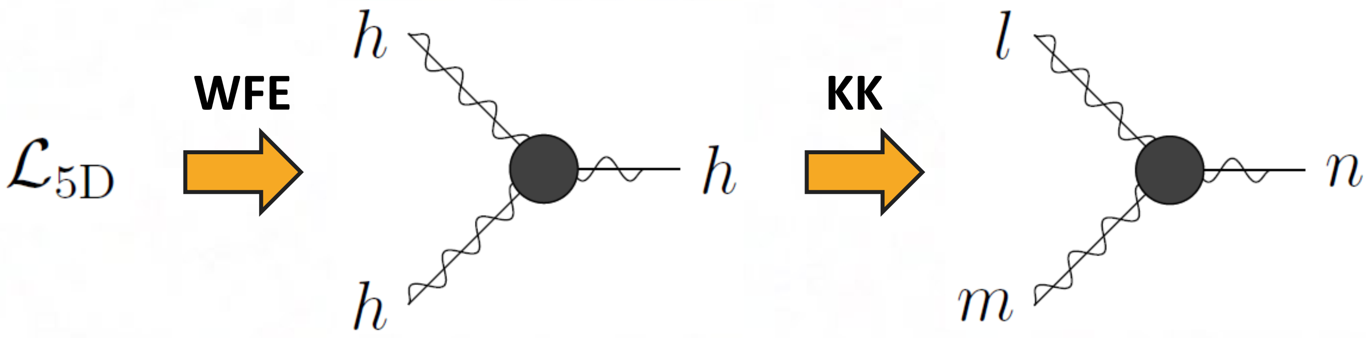
$$\mathcal{L}_{5\text{D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} \xrightarrow[\text{\& KK}]{\text{WFE}} \mathcal{L}_{4\text{D}}^{(\text{eff})} \equiv \int dy\ \ \mathcal{L}_{5\text{D}}$$

Example: KK Mode Interactions (& Couplings)

$\mathcal{L}_{5D} \xrightarrow{\textbf{WFE}}$ Interactions between h(x,y) & r(x) $\xrightarrow{\textbf{KK}}$ Interactions between 4D states

$\mathcal{M}_{\rm full} =$ 

Example: Cubic KK Mode Interaction



If **two spatial** derivatives...

If **two extra-dimensional** derivatives...

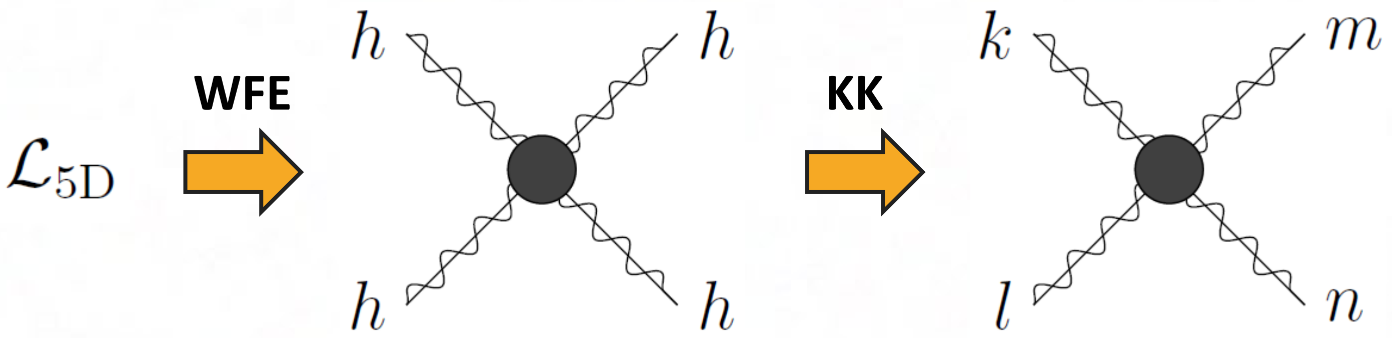
A-Type Coupling

$$a_{lmn} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-2}\, \psi_l\, \psi_m\, \psi_n$$

B-Type Coupling

$$b_{l'm'n} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-4}\, (\partial_\varphi \psi_l)\,(\partial_\varphi \psi_m)\, \psi_n$$

Example: Quartic KK Mode Interaction



If **two spatial** derivatives...

If **two extra-dimensional** derivatives...

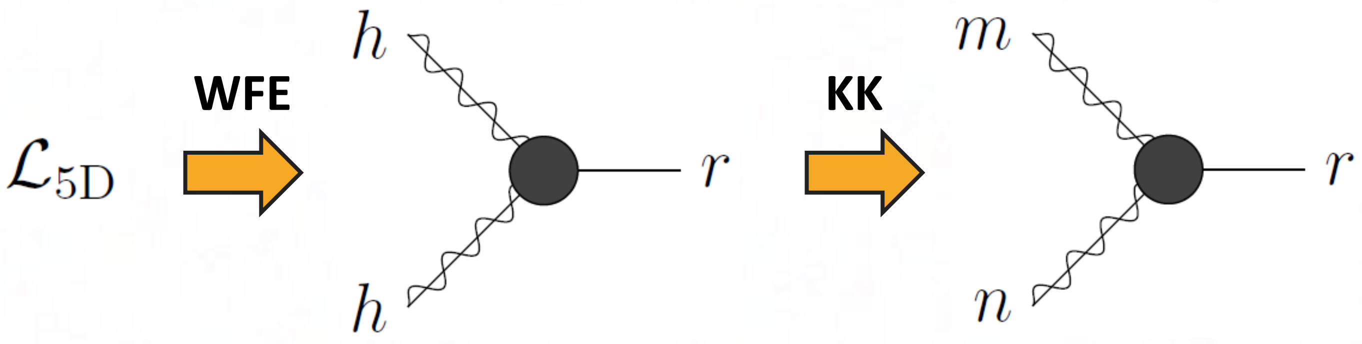
A-Type Coupling

$$a_{klmn} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-2}\, \psi_k\, \psi_l\, \psi_m\, \psi_n$$

B-Type Coupling

$$b_{k'l'mn} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-4}\, (\partial_\varphi \psi_k)\, (\partial_\varphi \psi_l)\, \psi_m\, \psi_n$$

Example: (KK Mode)^2 - Radion Interaction



A-Type forbidden by
4D diffeomorphism invariance

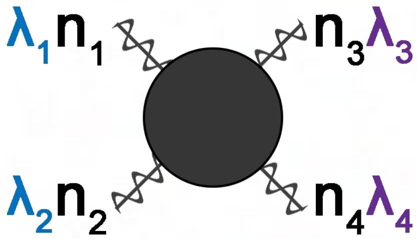
If two extra-dimensional derivatives...

B-Type Coupling

$$b_{m'n'r} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-4}\,(\partial_\varphi \psi_m)\,(\partial_\varphi \psi_n)\,\Big[\underbrace{\varepsilon^{+2} e^{-\pi k r_c} \psi_0}_{\text{troublesome}}\Big]$$

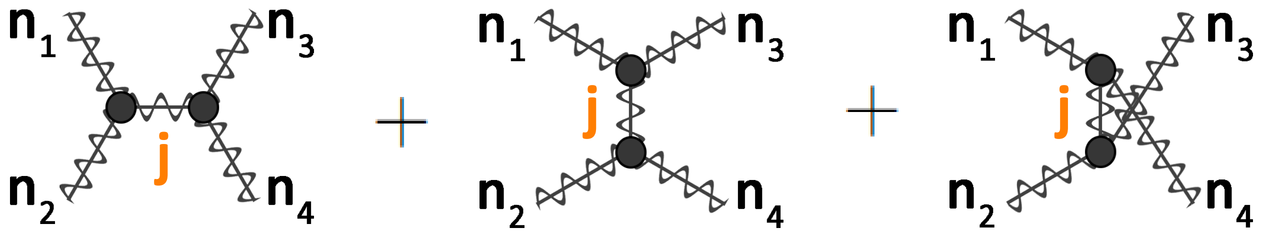
Matrix Element: Total Matrix Element

$$\mathcal{M}_{\text{full}} =$$

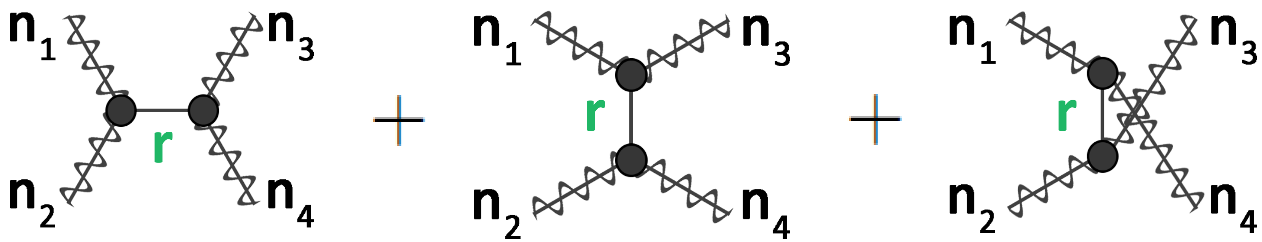


$$= \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{+\infty} \mathcal{M}_j$$

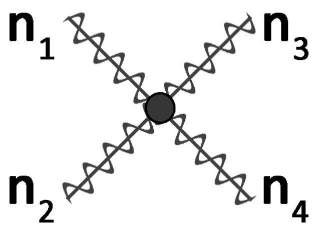
$$\mathcal{M}_j =$$



$$\mathcal{M}_r =$$



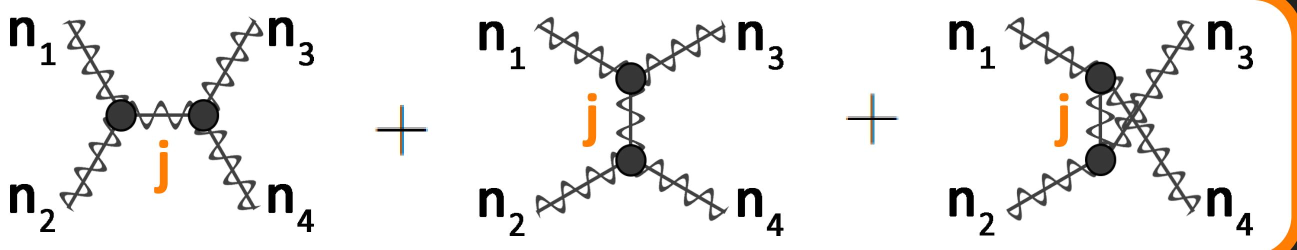
$$\mathcal{M}_c =$$



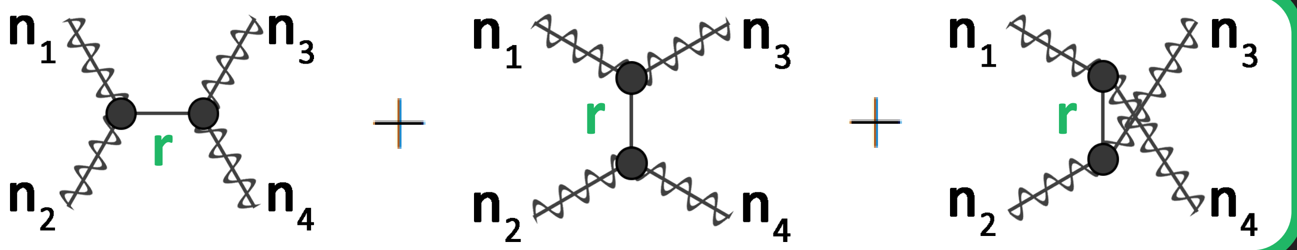
Matrix Element: Truncated Matrix Element

$$\mathcal{M}^{[\underline{N}]} \equiv \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{\underline{N}} \mathcal{M}_j$$

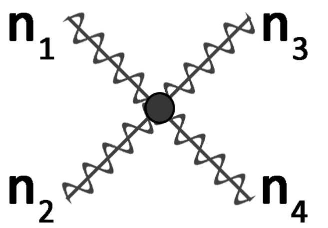
$\mathcal{M}_j =$



$\mathcal{M}_r =$



$\mathcal{M}_c =$



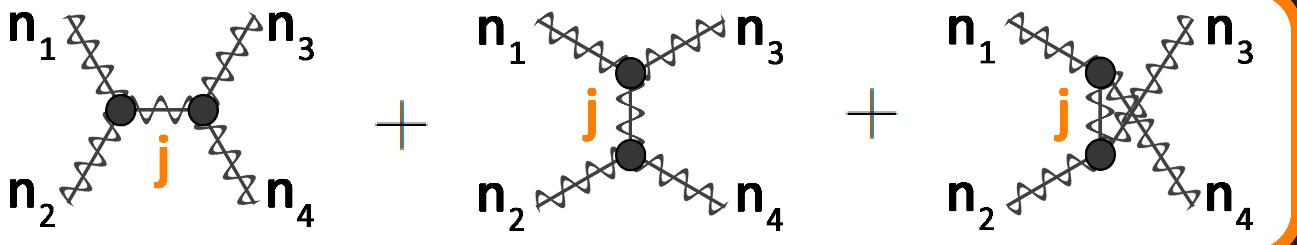
Matrix Element: Expanded Matrix Element

$$\mathcal{M}^{[N]} \equiv \sum_{\sigma} \underline{\mathcal{M}^{[N](\sigma)}} \equiv \sum_{\sigma} \underline{\overline{\mathcal{M}}^{[N](\sigma)}} \; (E r_c)^{\underline{2\sigma}}$$

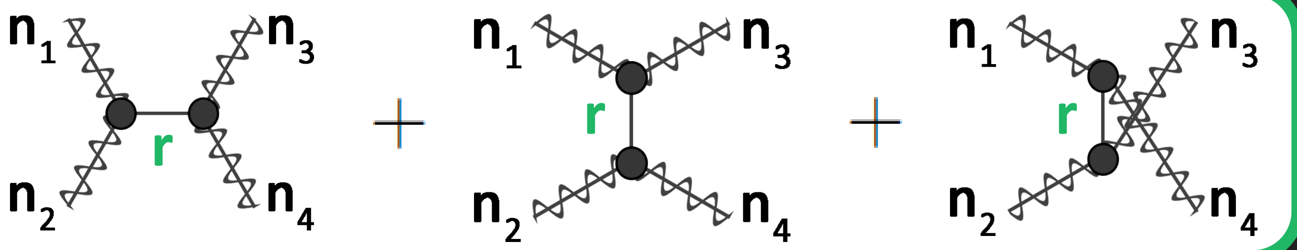
From 5D Theory:

$\mathcal{M}^{[N](\sigma)}$ vanishes as $N \to \infty$ for all $\sigma > 1$ (i.e. growth $> E^2$)

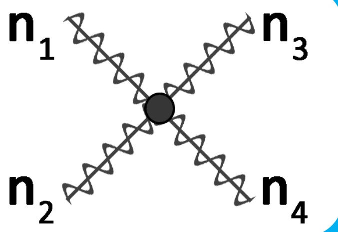
$\mathcal{M}_j =$



$\mathcal{M}_r =$



$\mathcal{M}_c =$



Some Obstacles

- **Weak Field Expansion** is algebraically complicated and computationally intensive (many indices, thousands of terms)

- The resulting **4D Interactions** are long even before KK decomposition.

- Couplings involve integrating products of highly-oscillatory wavefunctions (**numerically unstable**) but require **high precision** for evidence of cancellations.

$$
\begin{aligned}
\overline{\mathcal{L}}_{A:hhhh} = &\ \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\partial^\nu\hat{h}^{\rho\sigma}) - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\sigma\tau}(\partial^\mu\partial^\nu\hat{h}^{\rho}_{\tau}) \\
& - \frac{3}{4}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\sigma})(\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\sigma\tau}(\partial^\mu\partial^\rho\hat{h}^{\nu}_{\tau}) \\
& - \frac{1}{4}\llbracket\hat{h}\hat{h}\rrbracket\hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial^\nu\hat{h}^{\sigma}_{\tau}) \\
& + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\nu\tau})(\partial^\rho\hat{h}^{\sigma}_{\tau}) + \frac{1}{8}\hat{h}^2\hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) \\
& - \frac{1}{8}\hat{h}^2\hat{h}^{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\nu\tau})(\partial_\tau\hat{h}^{\rho\sigma}) \\
& + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial_\tau\hat{h}^{\tau\rho})(\partial^\mu\hat{h}^{\nu\sigma}) + \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{h}^{\nu\rho})(\partial_\rho\hat{h}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\partial^\nu\partial^\sigma\hat{h}) + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\partial^\nu\partial_\tau\hat{h}^{\tau\sigma}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial_\tau\hat{h}^{\nu\sigma}) + \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\partial_\rho\hat{h}) \\
& - \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\partial^\sigma\hat{h}_{\sigma\rho}) + \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{h}_{\rho\sigma})(\partial^\rho\hat{h}^{\nu\sigma}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial^\sigma\hat{h}^{\nu}_{\tau}) - \hat{h}_{\mu\nu}\hat{h}^{\nu\rho}\hat{h}_{\sigma\tau}(\partial^\mu\partial_\rho\hat{h}^{\sigma\tau}) \\
& - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\hat{h}_{\sigma\tau})(\partial_\rho\hat{h}^{\sigma\tau}) + \hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\Box\hat{h}^{\mu}_{\rho}) \\
& + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) - \frac{1}{4}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\tau\hat{h}^{\mu\nu})(\partial_\tau\hat{h}^{\rho\sigma}) \\
& - \frac{1}{16}\hat{h}^2\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{8}\hat{h}\hat{h}_{\mu\nu}(\partial^\sigma\hat{h}^{\mu\nu})(\partial_\sigma\hat{h}) \\
& - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\Box\hat{h}^{\nu\sigma}) - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\hat{h}_{\rho\sigma})(\partial^\sigma\hat{h}) \\
& + \frac{3}{2}\hat{h}\hat{h}_{\mu\nu}(\partial_\rho\hat{h}^{\mu\sigma})(\partial^\rho\hat{h}^{\nu}_{\sigma}) + \frac{1}{48}\hat{h}^3(\Box\hat{h}) \\
& + \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\rho\hat{h}_{\rho\sigma})(\partial^\sigma\hat{h}^{\mu\nu}) + \frac{1}{4}\hat{h}\llbracket\hat{h}\hat{h}\rrbracket(\partial^\mu\partial^\nu\hat{h}_{\mu\nu}) \\
& - \frac{1}{8}\hat{h}\llbracket\hat{h}\hat{h}\rrbracket(\Box\hat{h})\ .
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathcal{L}}_{B:hhhh} = &\ \frac{1}{2}\llbracket\hat{h}\hat{h}'\rrbracket^2 - \frac{1}{2}\hat{h}\llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}'\rrbracket - \frac{1}{2}\llbracket\hat{h}\hat{h}'\hat{h}\hat{h}'\rrbracket \\
& + \frac{1}{2}\hat{h}\llbracket\hat{h}\hat{h}'\hat{h}'\rrbracket + \llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}\hat{h}'\rrbracket - \llbracket\hat{h}\hat{h}\hat{h}'\hat{h}'\rrbracket \\
& + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\hat{h}'\rrbracket - \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\rrbracket^2 - \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\hat{h}'\rrbracket \\
& + \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\rrbracket^2\ ,
\end{aligned}
$$

If 4D is trouble, why do it?

- The calculations relevant to experimental quantities rely on the **4D particle content**, e.g. matrix elements and cross-sections.

- Although the 4D theory must be consistent with the 5D theory, how that is achieved is not obvious: **nontrivial relationships b/w KK mode masses, couplings.**

- Facilitates **better understanding** of how to efficiently calculate quantities involving massive spin-2 particles (and the **development of automated tools**!)

$$
\begin{aligned}
\overline{\mathcal{L}}_{A:hhhh} = {} & \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\partial^\nu\hat{h}^{\rho\sigma}) - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\sigma\tau}(\partial^\mu\partial^\nu\hat{h}^{\rho}_{\tau}) \\
& - \frac{3}{4}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\sigma})(\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\sigma\tau}(\partial^\mu\partial^\rho\hat{h}^{\nu}_{\tau}) \\
& - \frac{1}{4}\llbracket\hat{h}\hat{h}\rrbracket\hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial^\nu\hat{h}^{\sigma}_{\tau}) \\
& + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\nu\tau})(\partial^\rho\hat{h}^{\sigma}_{\tau}) + \frac{1}{8}\hat{h}^2\hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) \\
& - \frac{1}{8}\hat{h}^2\hat{h}^{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\nu\tau})(\partial_\tau\hat{h}^{\rho\sigma}) \\
& + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial_\tau\hat{h}^{\tau\rho})(\partial^\mu\hat{h}^{\nu\sigma}) + \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{h}^{\nu\rho})(\partial_\rho\hat{h}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\partial^\nu\partial^\sigma\hat{h}) + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\partial^\nu\partial_\tau\hat{h}^{\tau\sigma}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial_\tau\hat{h}^{\nu\sigma}) + \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\partial_\rho\hat{h}) \\
& - \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\partial^\sigma\hat{h}_{\sigma\rho}) + \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{h}_{\rho\sigma})(\partial^\rho\hat{h}^{\nu\sigma}) \\
& - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h}^{\rho\tau})(\partial^\sigma\hat{h}^{\nu}_{\tau}) - \hat{h}_{\mu\nu}\hat{h}^{\nu\rho}\hat{h}_{\sigma\tau}(\partial^\mu\partial_\rho\hat{h}^{\sigma\tau}) \\
& - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\hat{h}_{\sigma\tau})(\partial_\rho\hat{h}^{\sigma\tau}) + \hat{h}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\Box\hat{h}^{\mu}_{\rho}) \\
& + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) - \frac{1}{4}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\tau\hat{h}^{\mu\nu})(\partial_\tau\hat{h}^{\rho\sigma}) \\
& - \frac{1}{16}\hat{h}^2\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{8}\hat{h}\hat{h}_{\mu\nu}(\partial^\sigma\hat{h}^{\mu\nu})(\partial_\sigma\hat{h}) \\
& - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}\hat{h}^{\mu\rho}(\Box\hat{h}^{\nu\sigma}) - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}^{\nu\rho}(\partial^\mu\hat{h}_{\rho\sigma})(\partial^\sigma\hat{h}) \\
& + \frac{3}{2}\hat{h}\hat{h}_{\mu\nu}(\partial_\rho\hat{h}^{\mu\sigma})(\partial^\rho\hat{h}^{\nu}_{\sigma}) + \frac{1}{48}\hat{h}^3(\Box\hat{h}) \\
& + \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\rho\hat{h}_{\rho\sigma})(\partial^\sigma\hat{h}^{\mu\nu}) + \frac{1}{4}\hat{h}\llbracket\hat{h}\hat{h}\rrbracket(\partial^\mu\partial^\nu\hat{h}_{\mu\nu}) \\
& - \frac{1}{8}\hat{h}\llbracket\hat{h}\hat{h}\rrbracket(\Box\hat{h})\ . \\
\overline{\mathcal{L}}_{B:hhhh} = {} & \frac{1}{2}\llbracket\hat{h}\hat{h}'\rrbracket^2 - \frac{1}{2}\hat{h}\llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}'\rrbracket - \frac{1}{2}\llbracket\hat{h}\hat{h}'\hat{h}\hat{h}'\rrbracket \\
& + \frac{1}{2}\hat{h}\llbracket\hat{h}\hat{h}'\hat{h}'\rrbracket + \llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}\hat{h}'\rrbracket - \llbracket\hat{h}\hat{h}\hat{h}'\hat{h}'\rrbracket \\
& + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\hat{h}'\rrbracket - \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\rrbracket^2 - \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\hat{h}'\rrbracket \\
& + \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\rrbracket^2\ ,
\end{aligned}
$$

Cancelling Bad High-Energy Growth

Sum Rules & Numerical Checks

KK Decomposition: Randall-Sundrum 1

$$\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}^{(n)}_{\mu\nu}(x)\, \underline{\psi_n(\varphi)}$$

$$\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}(x)\, \underline{\psi_0}$$

Boundary Condition

$$\left(\partial_\varphi \psi_n\right)\Big|_{\varphi=0} = \left(\partial_\varphi \psi_n\right)\Big|_{\varphi=\pm\pi} = 0$$

Sturm-Liouville Equation

$$\partial_\varphi \left[\varepsilon^{-4}(\partial_\varphi \psi_n)\right] = -(m_n r_c)^2 \varepsilon^{-2} \psi_n$$

Solutions

$$\psi_n$$

Normalization

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad e^{-2kr_c|\varphi|}\psi_m \psi_n = \delta_{m,n}$$

B → A

Sturm-Liouville Equation

$$\partial_\varphi\left[\varepsilon^{-4}(\partial_\varphi\psi_n)\right] = -(m_n r_c)^2\varepsilon^{-2}\psi_n$$

$$\boxed{\mu_n \equiv m_n r_c}$$

B → A

Sturm-Liouville Equation

$$\partial_\varphi \left[\varepsilon^{-4}(\partial_\varphi \psi_n)\right] = -(m_n r_c)^2 \varepsilon^{-2} \psi_n$$

$$\mu_n \equiv m_n r_c$$

$$\begin{aligned}
\mu_n^2 a_{mn} &\equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-2}\psi_m \left[\mu_n^2 \psi_n\right] \\
&\overset{\text{SL Eq.}}{=} \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-4}\psi_m \left[-\varepsilon^{+2}\partial_\varphi \varepsilon^{-4}(\partial_\varphi \psi_n)\right] \\
&\overset{\text{IBP}}{=} \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-4}(\partial_\varphi \psi_m)(\partial_\varphi \psi_n) \equiv b_{m'n'}
\end{aligned}$$

B → A

Sturm-Liouville Equation

$$\partial_\varphi\left[\varepsilon^{-4}(\partial_\varphi\psi_n)\right] = -(m_n r_c)^2\varepsilon^{-2}\psi_n$$

$$\mu_n \equiv m_n r_c$$

$$\begin{aligned}
\mu_n^2 a_{mn} &\equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-2}\psi_m\left[\mu_n^2\psi_n\right] \\
&\overset{\text{SL Eq.}}{=} \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-4}\psi_m\left[-\varepsilon^{+2}\partial_\varphi\varepsilon^{-4}(\partial_\varphi\psi_n)\right] \\
&\overset{\text{IBP}}{=} \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_m)(\partial_\varphi\psi_n) \equiv b_{m'n'}
\end{aligned}$$

$$b_{l'm'n} = \frac{1}{2}\left[\mu_l^2 + \mu_m^2 - \mu_n^2\right] a_{lmn}$$

$$b_{k'l'mn} = \frac{1}{6}\left[2\mu_k^2 + 2\mu_l^2 - \mu_m^2 - \mu_n^2\right] a_{klmn}$$

B → A

Sturm-Liouville Equation

$$\partial_\varphi\left[\varepsilon^{-4}(\partial_\varphi\psi_n)\right]=-(m_n r_c)^2\varepsilon^{-2}\psi_n$$

$$\mu_n\equiv m_n r_c$$

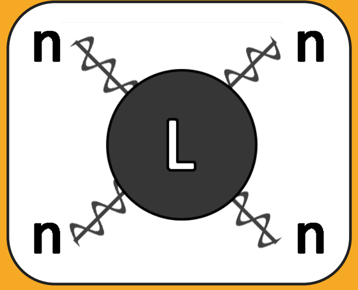
Contrast with the radion coupling…

$$b_{m'n'r}\equiv\frac{1}{\pi}\int_{-\pi}^{+\pi}d\varphi\quad\varepsilon^{-4}\;(\partial_\varphi\psi_m)\,(\partial_\varphi\psi_n)\,\underbrace{\left[\varepsilon^{+2}e^{-\pi k r_c}\psi_0\right]}_{\text{troublesome}}$$

$$b_{l'm'n}=\frac{1}{2}\left[\mu_l^2+\mu_m^2-\mu_n^2\right]a_{lmn}$$

$$b_{k'l'mn}=\frac{1}{6}\left[2\mu_k^2+2\mu_l^2-\mu_m^2-\mu_n^2\right]a_{klmn}$$

Helicity-Zero Elastic Matrix Element

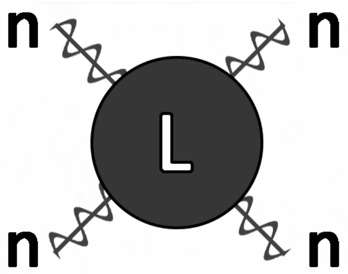


Consider **elastic** = all equal KK numbers

Focus on the **fastest-growing helicity combo***

All External States = Helicity 0

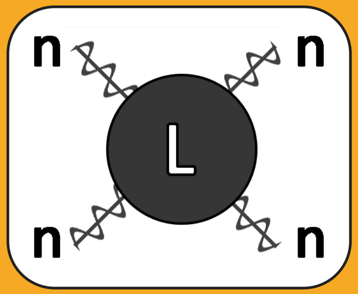
$$\mathcal{M}_{\text{full}} =$$



$$= \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{+\infty} \mathcal{M}_j \quad \sim \mathcal{O}(s^5)$$

* before applying sum rules

Helicity-Zero Elastic Matrix Element



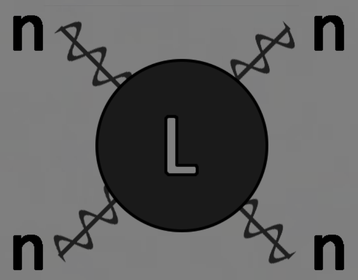
Process Indicator

Consider **elastic** = all equal KK numbers

Focus on the **fastest-growing helicity combo***

All External States = Helicity 0

$\mathcal{M}_{\text{full}} =$



$$= \mathcal{M}_c + \mathcal{M}_r + \sum_{j=0}^{+\infty} \mathcal{M}_j \sim \mathcal{O}(s^5)$$

* before applying sum rules

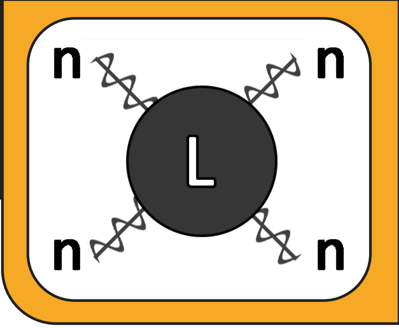
O(s^5): Must Vanish

$$\underline{\overline{\mathcal{M}}_c^{(5)}} = -\frac{\kappa^2\, a_{nnnn}}{2304\,\pi r_c\, m_n^8}\,[7+\cos(2\theta)]\,\sin^2\theta$$

$$\underline{\overline{\mathcal{M}}_j^{(5)}} = \frac{\kappa^2\, a_{nnj}^2}{2304\,\pi r_c\, m_n^8}\,[7+\cos(2\theta)]\,\sin^2\theta$$

$$\boxed{\overline{\mathcal{M}}^{(5)} = \frac{\kappa^2\,[7+\cos(2\theta)]\,\sin^2\theta}{2304\,\pi r_c\, m_n^8}\left\{\sum_{j=0}^{+\infty} a_{nnj}^2 - a_{nnnn}\right\}}$$

O(s^4): Must Vanish

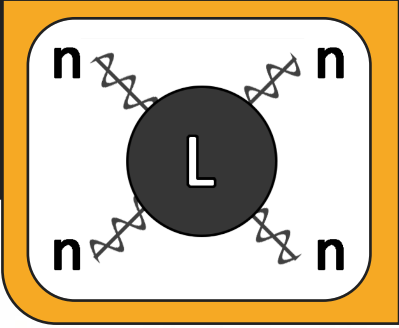


$$\overline{\mathcal{M}}_c^{(4)} = \frac{\kappa^2\, a_{nnnn}}{6912\,\pi r_c\, m_n^6}\left[63 - 196\cos(2\theta) + 5\cos(4\theta)\right]$$

$$\overline{\mathcal{M}}_j^{(4)} = -\frac{\kappa^2\, a_{nnj}^2}{9216\,\pi r_c\, m_n^6}\left\{\left[7+\cos(2\theta)\right]^2\frac{m_j^2}{m_n^2} + 2\left[9 - 140\cos(2\theta) + 3\cos(4\theta)\right]\right\}$$

$$\boxed{\overline{\mathcal{M}}^{(4)} = \frac{\kappa^2\left[7+\cos(2\theta)\right]^2}{9216\,\pi r_c\, m_n^6}\left\{\frac{4}{3}a_{nnnn} - \sum_j \frac{m_j^2}{m_n^2}a_{nnj}^2\right\}}$$

O(s^3): Must Vanish



$$\overline{\mathcal{M}}^{(3)}_{c} = \frac{\kappa^2\, a_{nnnn}}{3456\,\pi r_c\, m_n^4}\left[-185 + 692\cos(2\theta) + 5\cos(4\theta)\right]$$

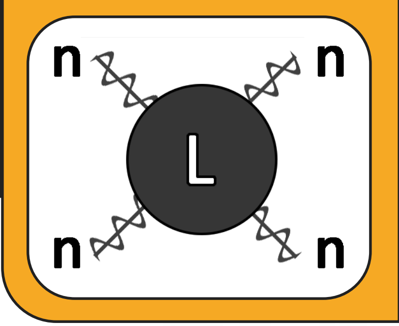
$$\overline{\mathcal{M}}^{(3)}_{r} = -\frac{\kappa^2}{32\,\pi r_c\, m_n^4}\left[\frac{b_{nnr}^2}{(m_n r_c)^4}\right]\sin^2\theta$$

$$\overline{\mathcal{M}}^{(3)}_{0} = \frac{\kappa^2\, a_{nn0}^2}{1152\,\pi r_c\, m_n^4}\left[15 - 270\cos(2\theta) - \cos(4\theta)\right]$$

$$\overline{\mathcal{M}}^{(3)}_{j>0} = \frac{\kappa^2\, a_{nnj}^2}{2304\,\pi r_c\, m_n^4}\left\{5\left[1-\cos(2\theta)\right]\frac{m_j^4}{m_n^4} + \left[69 + 60\cos(2\theta) - \cos(4\theta)\right]\frac{m_j^2}{m_n^2} + 2\left[13 - 268\cos(2\theta) - \cos(4\theta)\right]\right\}$$

$$\boxed{\overline{\mathcal{M}}^{(3)} = \frac{5\,\kappa^2\sin^2\theta}{1152\,\pi r_c\, m_n^4}\left\{\sum_j \frac{m_j^4}{m_n^4}a_{nnj}^2 - \frac{16}{15}a_{nnnn} - \frac{4}{5}\left[\frac{9\,b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]\right\}}$$

O(s^2): Must Vanish



$$\overline{\mathcal{M}}_c^{(2)} = -\frac{\kappa^2\, a_{nnnn}}{54\,\pi r_c\, m_n^2}\left[5 + 47\cos(2\theta)\right]$$

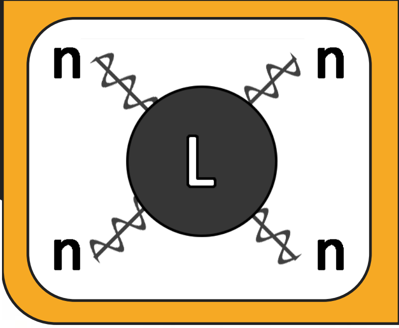
$$\overline{\mathcal{M}}_r^{(2)} = \frac{\kappa^2}{48\,\pi r_c\, m_n^2}\left[\frac{b_{nnr}^2}{(m_n r_c)^4}\right]\left[7 + \cos(2\theta)\right]$$

$$\overline{\mathcal{M}}_0^{(2)} = \frac{\kappa^2\, a_{nn0}^2}{576\,\pi r_c\, m_n^2}\left[175 + 624\cos(2\theta) + \cos(4\theta)\right]$$

$$\overline{\mathcal{M}}_{j>0}^{(2)} = \frac{\kappa^2\, a_{nnj}^2}{6912\,\pi r_c\, m_n^2}\Bigg\{4\left[7 + \cos(2\theta)\right]\left[5 - 2\frac{m_j^2}{m_n^2}\right]\frac{m_j^4}{m_n^4} - \left[1291 + 1132\cos(2\theta) + 9\cos(4\theta)\right]\frac{m_j^2}{m_n^2}$$
$$+ 4\left[553 + 1876\cos(2\theta) + 3\cos(4\theta)\right]\Bigg\}$$

$$\boxed{\overline{\mathcal{M}}^{(2)} = \frac{\kappa^2\,\left[7 + \cos(2\theta)\right]}{864\,\pi r_c\, m_n^2}\left\{\sum_j \left[\frac{m_j^2}{m_n^2} - \frac{5}{2}\right]\frac{m_j^4}{m_n^4} a_{nnj}^2 + \frac{8}{3} a_{nnnn} - 2\left[\frac{9\, b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]\right\}}$$

Sum Rules: Summary (order-by-order)



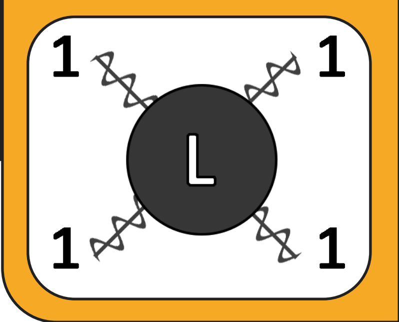
The **longitudinal elastic scattering matrix elements** grows like O(E^2) iff the following **sum rules** between masses & couplings hold true:

$$\underline{\mathcal{O}(s^5)} : \quad \sum_{j=0}^{+\infty} a_{nnj}^2 = a_{nnnn}$$

$$\underline{\mathcal{O}(s^4)} : \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^2 a_{nnj}^2 = \frac{4}{3} a_{nnnn}$$

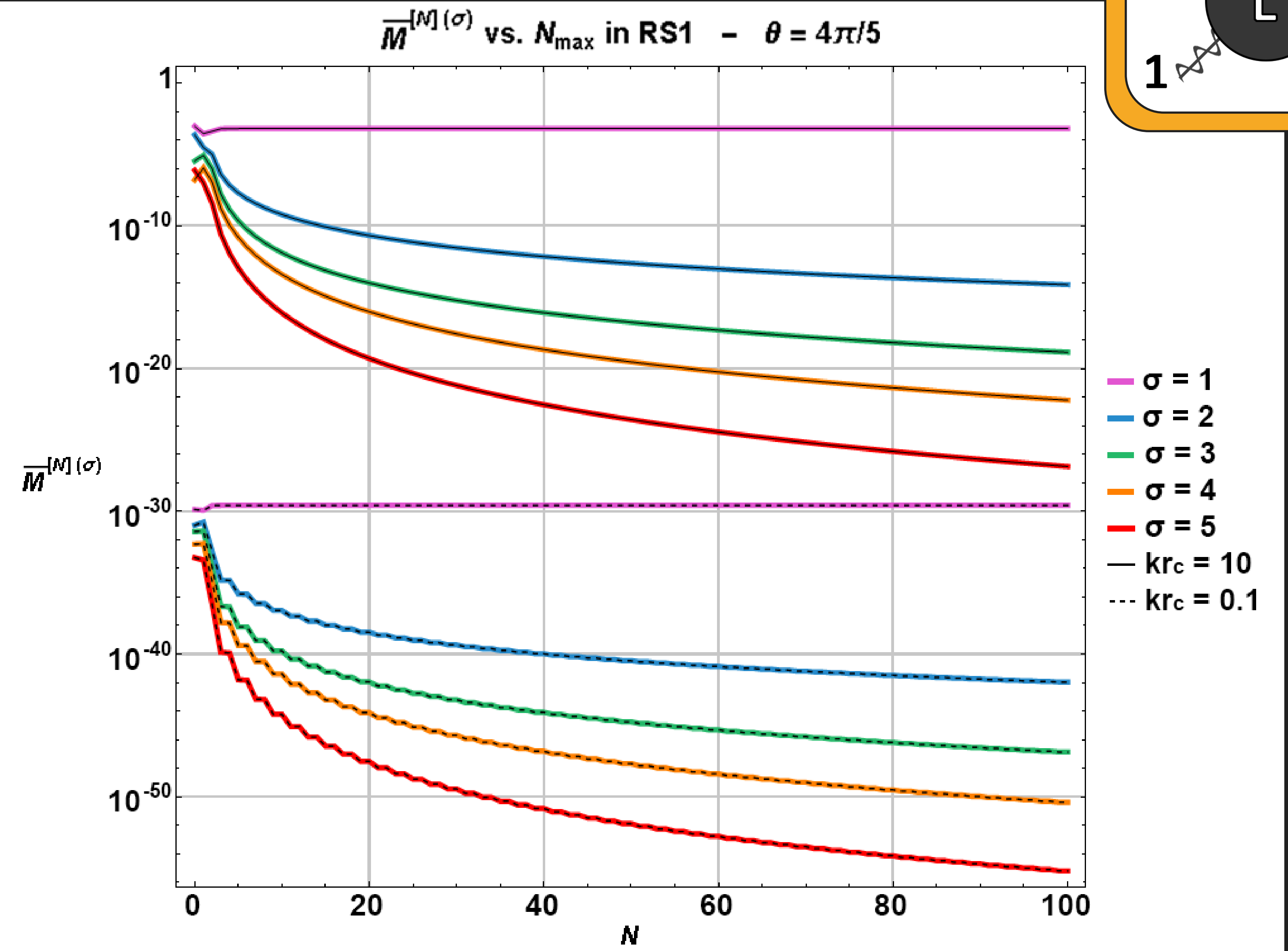
$$\underline{\mathcal{O}(s^3)} : \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = \frac{4}{5}\left[9\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right] + \frac{16}{15} a_{nnnn}$$

$$\underline{\mathcal{O}(s^2)} : \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^6 a_{nnj}^2 = 4\left[9\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]$$



$$\mathcal{M}^{[N]} = \sum_{\sigma} \overline{\mathcal{M}}^{[N](\sigma)} \; (Er_c)^{2\sigma}$$

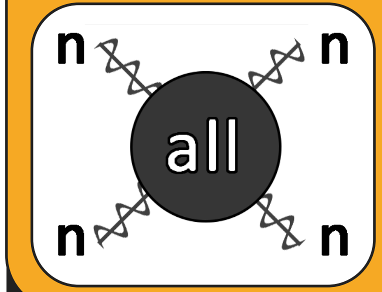
$$\mathcal{M}^{[N]} \equiv \; (\text{1st } N \text{ KK Modes})$$



Q: How does including more terms in the truncated sum affect leading energy growth?

A: By increasing N, any energy growth faster than $O(E^2)$ increasingly cancels. $O(E^2)$ remains.

Elastic 2-to-2 KK Mode Scattering Matrix Elements in RS1
Fastest Energy Growth per Helicity Combination: $(\lambda_1,\lambda_2) \rightarrow (\lambda_3,\lambda_4)$



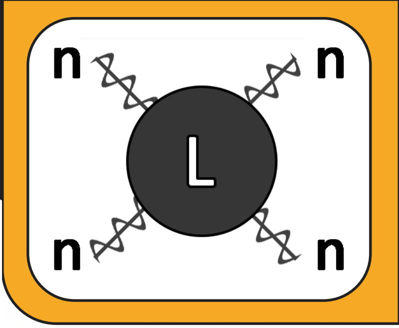
λ_1	λ_2: -2					-1					0					+1					+2				
-2	2	-1	0	-3	-2	-1	2	1	0	-1	0	1	2	1	0	-3	0	1	2	1	-2	-1	0	1	2
	-1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	-1	2	1	0	-1	0	1	2	1
	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0
	-3	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1
	-2	-3	-4	-5	-6	-1	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2
-1	-1	2	1	0	-1	0	1	2	-1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1
	0	-1	-2	-3	-4	-1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	-1	0
	-1	-2	-3	-4	-5	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-3
0	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0
	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1
	0	-1	-2	-3	-4	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0
+1	-3	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-1
	0	-1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	-1	-4	-3	-2	-1	0
	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1
	2	1	0	-1	-2	1	2	1	0	-1	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2
	1	0	-1	-2	-3	2	1	0	-1	-2	1	2	1	0	-1	0	-1	2	1	0	-1	0	1	2	-1
+2	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-1	-6	-5	-4	-3	-2
	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0	-5	-4	-3	-2	-3
	0	1	2	1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	1	-4	-3	-2	-1	0
	1	2	1	0	-1	0	1	2	-1	0	-1	0	1	2	1	-2	-1	0	1	2	-3	-2	-1	0	-1
	2	1	0	-1	-2	1	2	1	0	-3	0	1	2	1	0	-1	0	1	2	-1	-2	-3	0	-1	2

λ_3 \ λ_4	-2	-1	0	+1	+2
-2	-2	-1	0	1	2
-1	-1	0	1	2	1
0	0	1	2	1	0
+1	1	2	1	0	-1
+2	2	1	0	-1	-2

Legend

X = X = O(E^X) growth

Sum Rules: Summary (order-by-order)



The **longitudinal elastic scattering matrix elements** grows like O(E^2) iff the following **sum rules** between masses & couplings hold true:

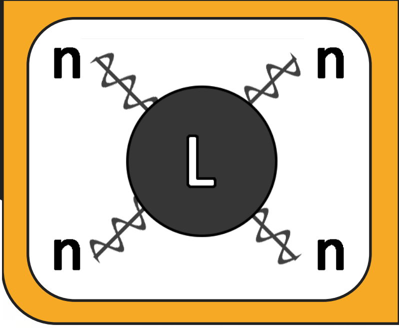
$$\underline{\mathcal{O}(s^5)}: \quad \sum_{j=0}^{+\infty} a_{nnj}^2 = a_{nnnn}$$ **PROVED**

$$\underline{\mathcal{O}(s^4)}: \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^2 a_{nnj}^2 = \frac{4}{3}\, a_{nnnn}$$ **PROVED**

$$\underline{\mathcal{O}(s^3)}: \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = \frac{4}{5}\left[9\,\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right] + \frac{16}{15}\, a_{nnnn}$$

$$\underline{\mathcal{O}(s^2)}: \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^6 a_{nnj}^2 = 4\left[9\,\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]$$

Sum Rules: Summary (order-by-order)



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Reorganize These Last 2

$$\mathcal{O}(s^3): \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = \frac{4}{5}\left[9\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right] + \frac{16}{15} a_{nnnn}$$

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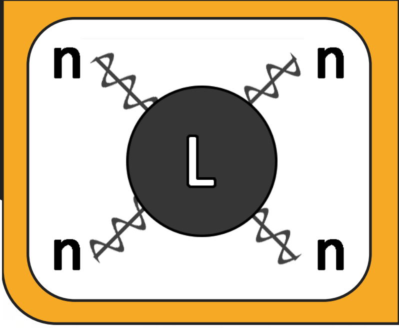
Sum Rules: Summary (alternate form)



The **longitudinal elastic scattering matrix elements** grows like O(E^2) iff the following **sum rules** between masses & couplings hold true:

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PROVED

$$\underline{\mathcal{O}(s^4)} : \quad \sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^2 a_{nnj}^2 = \frac{4}{3} a_{nnnn}$$

PROVED

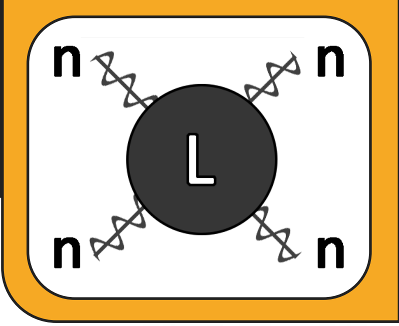
$$\underline{\mathcal{O}(s^2) \leftrightarrow \mathcal{O}(s^3)} : \quad \sum_{j=0}^{+\infty} \left[\left(\frac{m_j}{m_n}\right)^2 - 5\right] \left[\frac{m_j}{m_n}\right]^4 a_{nnj}^2 = -\frac{16}{3} a_{nnnn}$$

PROVED

$$\underline{\mathcal{O}(s^2)} : \quad \boxed{\sum_{j=0}^{+\infty} \left[\frac{m_j}{m_n}\right]^6 a_{nnj}^2 = 4\left[9\frac{b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]}$$

This is the only rule left without an analytic proof.

O(s): Actual Leading Energy Growth



$$\overline{\mathcal{M}}^{(1)}_c = \frac{\kappa^2\, a_{nnnn}}{1728\,\pi r_c}\left[1505 + 3108\cos(2\theta) - 5\cos(4\theta)\right]$$

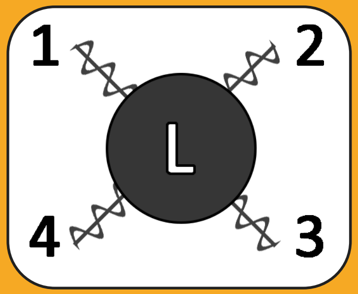
$$\overline{\mathcal{M}}^{(1)}_r = -\frac{\kappa^2}{24\,\pi r_c}\left[\frac{b^2_{nnr}}{(m_n r_c)^4}\right]\left[9 + 7\cos(2\theta)\right]$$

$$\overline{\mathcal{M}}^{(1)}_0 = \frac{\kappa^2\, a^2_{nn0}\,\csc^2\theta}{2304\,\pi r_c}\left[748 + 427\cos(2\theta) + 1132\cos(4\theta) - 3\cos(6\theta)\right]$$

$$\begin{aligned}\overline{\mathcal{M}}^{(1)}_{j>0} = \frac{\kappa^2\, a^2_{nnj}\,\csc^2\theta}{6912\,\pi r_c}\Bigg\{ & 3\left[7 + \cos(2\theta)\right]^2\frac{m_j^8}{m_n^8} - 4\left[241 + 148\cos(2\theta) - 5\cos(4\theta)\right]\frac{m_j^6}{m_n^6} \\ & + 4\left[787 + 604\cos(2\theta) - 47\cos(4\theta)\right]\frac{m_j^4}{m_n^4} - \left[3854 + 5267\cos(2\theta) + 98\cos(4\theta) - 3\cos(6\theta)\right]\frac{m_j^2}{m_n^2} \\ & + \left[2156 + 1313\cos(2\theta) + 3452\cos(4\theta) - 9\cos(6\theta)\right]\Bigg\}\end{aligned}$$

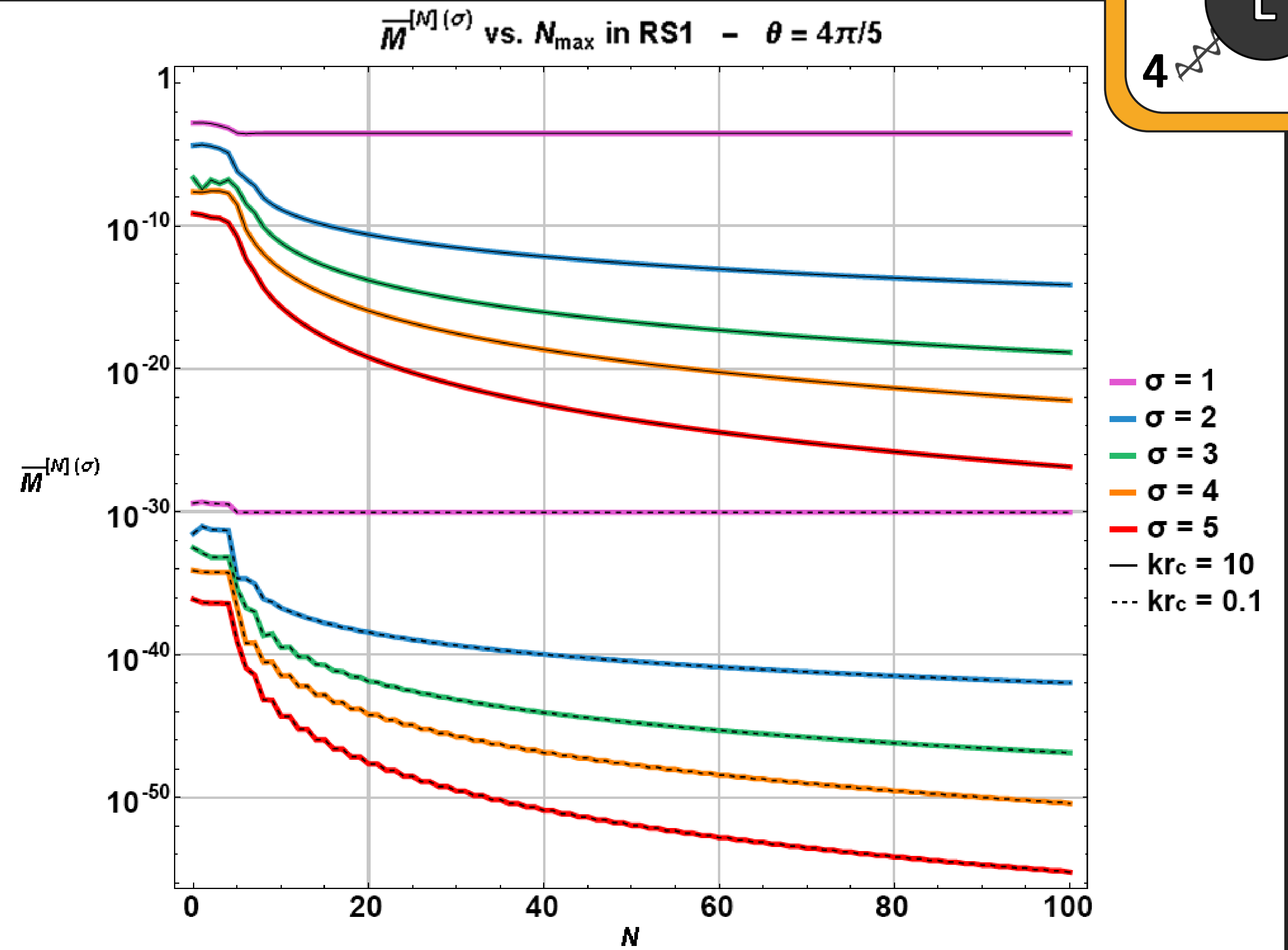
$$\boxed{\overline{\mathcal{M}}^{(1)} = \frac{\kappa^2\,\left[7 + \cos(2\theta)\right]^2\csc^2\theta}{2304\,\pi r_c}\left\{\sum_j \frac{m_j^8}{m_n^8}a^2_{nnj} + \frac{28}{15}a_{nnnn} - \frac{48}{5}\left[\frac{9\,b^2_{nnr}}{(m_n r_c)^4} - a^2_{nn0}\right]\right\}}$$

Longitudinal (1,4) → (2,3) : Cancellations



$$\mathcal{M}^{[N]} = \sum_{\sigma} \overline{\mathcal{M}}^{[N](\sigma)} \, (E r_c)^{2\sigma}$$

$$\mathcal{M}^{[N]} \equiv \text{(1st } N \text{ KK Modes)}$$



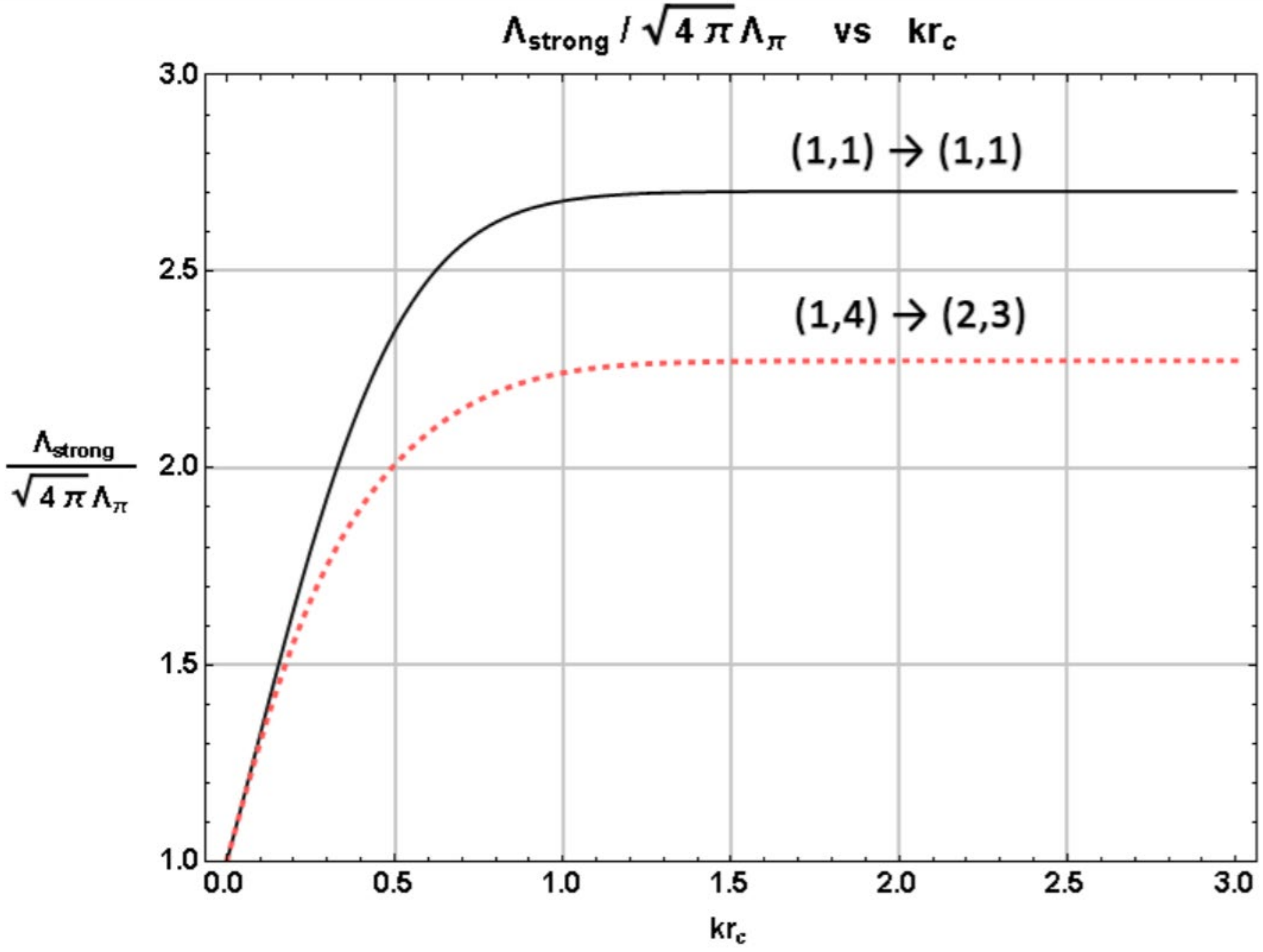
Q: How does including more terms in the truncated sum affect leading energy growth?

A: By increasing N, any energy growth faster than O(E^2) increasingly cancels. O(E^2) remains.

Strong Coupling Scale

Is it consistent with expectations?

Strong Coupling Scale



$$a^J = \frac{1}{32\pi^2}\int d\Omega \quad D^J_{\lambda_i\lambda_f}(\theta,\phi)\mathcal{M}(s,\theta,\phi)$$

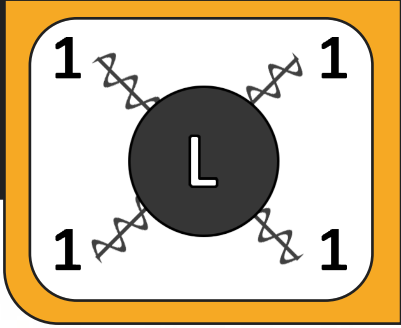
$$\sqrt{1-\frac{s_{\min}}{s}}\;\Re[a^J] \leq \frac{1}{2}$$

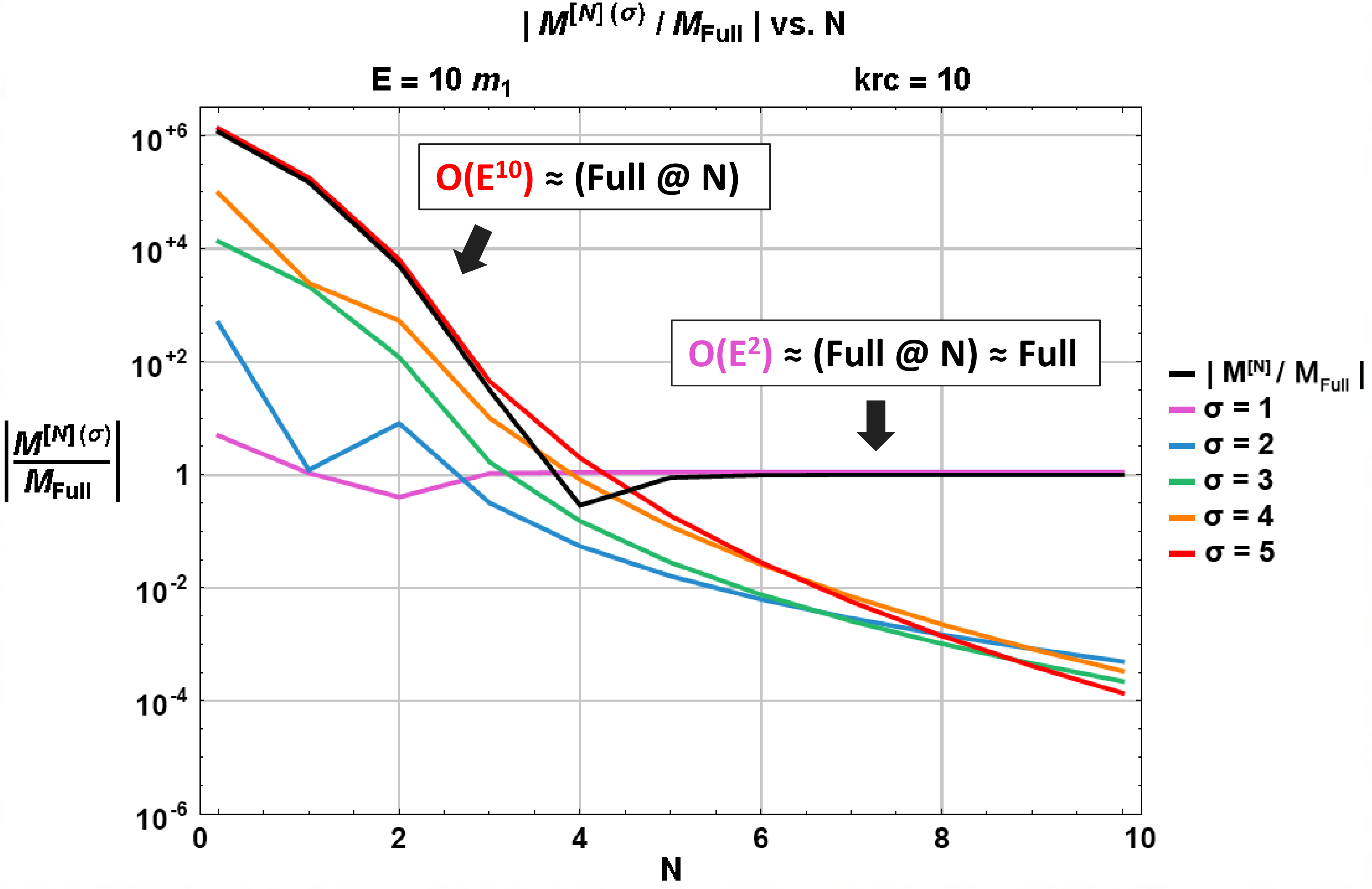
$$\Lambda_\pi \equiv M_{\text{Pl}}\, e^{-\pi k r_c}$$

Finite Truncation at Low Energy

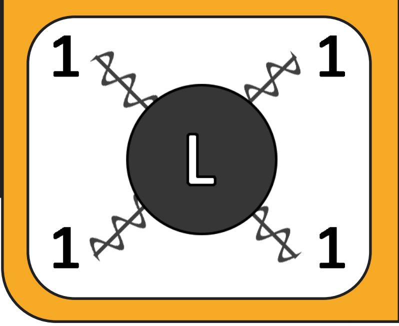
How many states should I include?

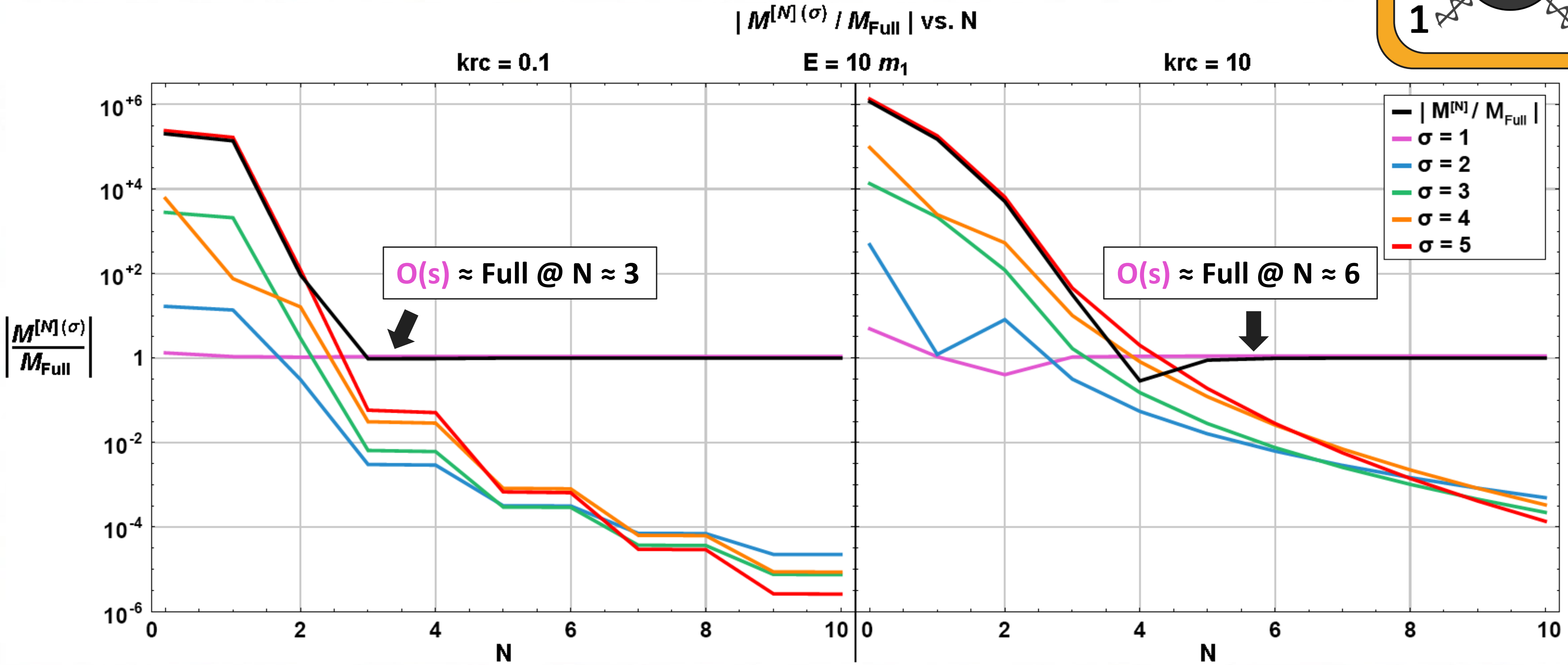
Truncation at Low Energies: $E = 10\ m_1$



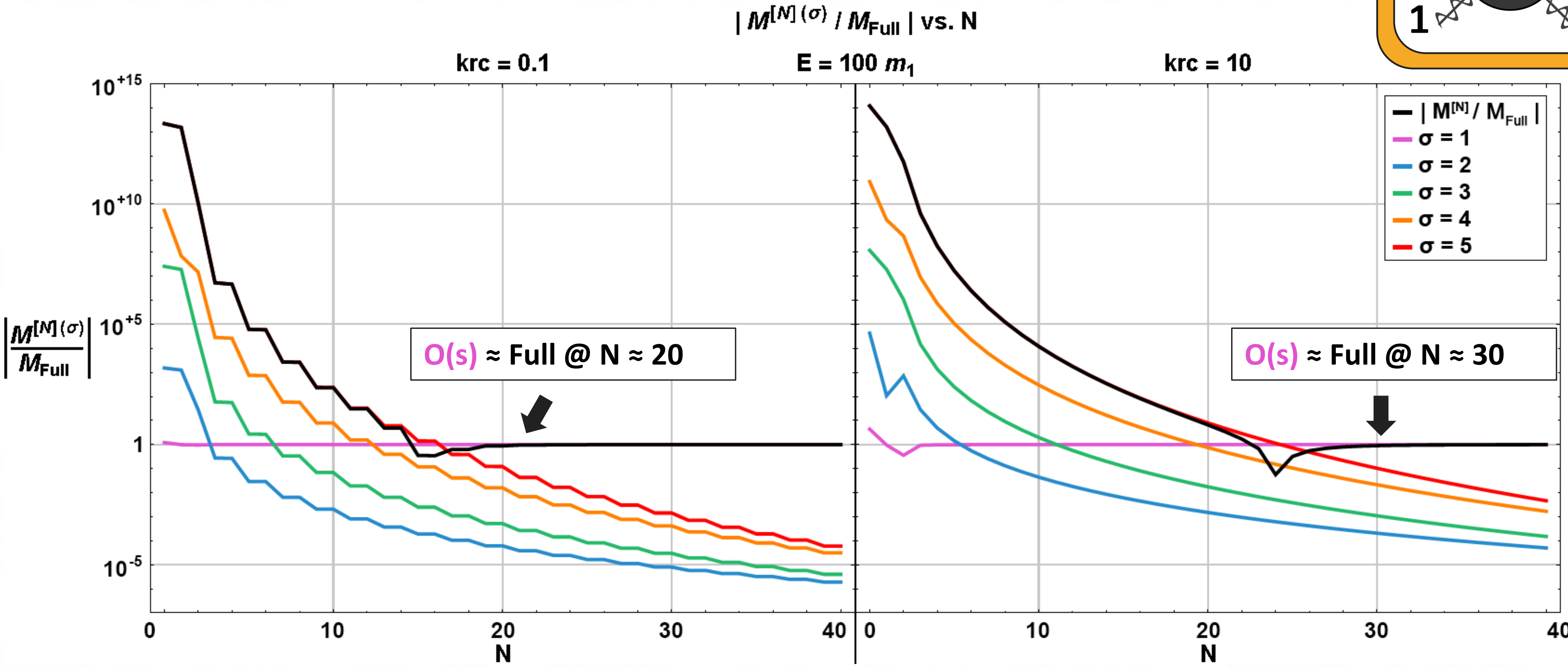


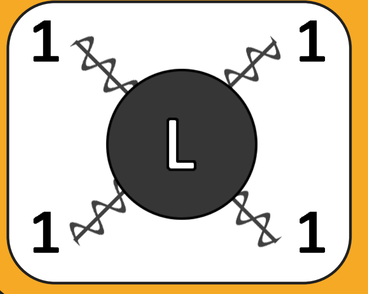
Truncation at Low Energies: $E = 10\ m_1$





Truncation at Low Energies: $E = 100\ m_1$

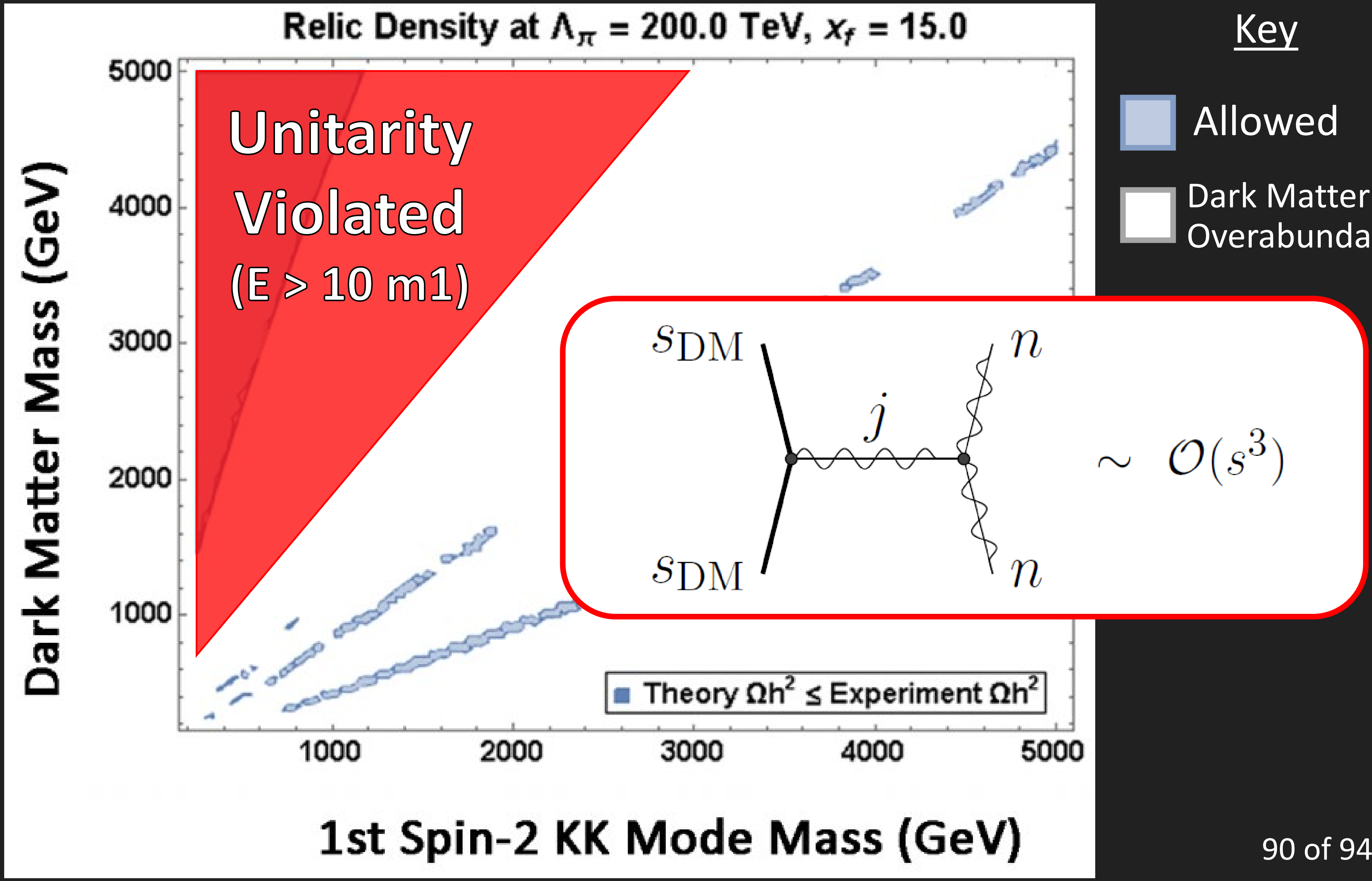




Late 2018

A ~~Mysterious~~ Region~~?~~

$$\Lambda_\pi \equiv M_{\rm Pl}\, e^{-\pi k r_c}$$



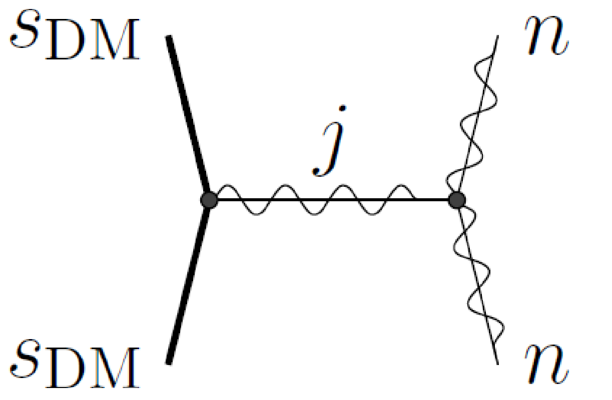
Key

Allowed

Dark Matter Overabundant

The Future
Bulk & Brane Matter; Other Extensions

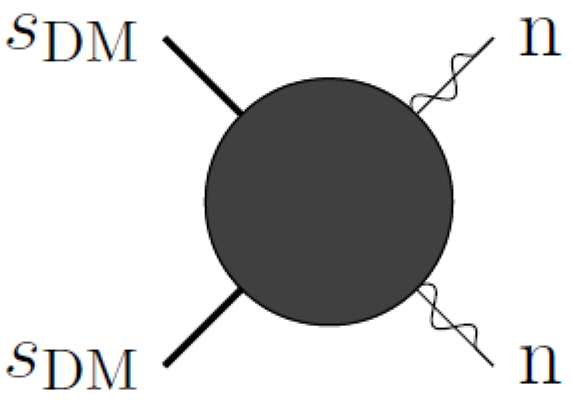
Future Work: Bulk & Brane Matter



$\sim \mathcal{O}(s^3)$

Troublesome DM Diagram

$\mathcal{M}_{\text{ann}} =$



$\sim \mathcal{O}(s)$

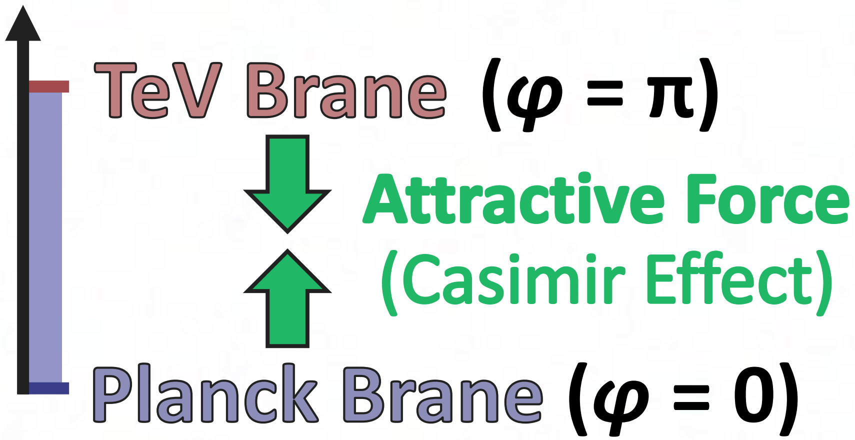
Proved for:

- Scalar Bulk Matter
- Scalar Brane Matter

<u>Also investigating:</u> bulk & brane matter of other spins, dark matter applications

Future Work: Radion Stabilization

- **4D Graviton:** measures fluctuations in the 4D directions.
- **Radion:** measures fluctuations in the separation between branes.



The radion requires *stabilization*
(or 5th dimension will collapse.)

- (Adding a **radion mass by hand** = $\mathcal{O}(s^2)$ divergence.)
- We're currently working on implementing a **fully stabilized theory.**

Conclusion

… and a few other applications which I hope to share with you soon!

- **Interested in learning more?**
  We have several papers available on the arXiv:
  - [arXiv:1906.11098] ; [arXiv:1910.06159] ; [arXiv: 2002.12458]

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Thank you for your time and attention!